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PROBLEMS OF DIFFRACTION AND PROPAGATION OF ELECTROMAGNETIC WAVES

by

V.A. Fok

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Block	Italic	Transliteration	Block	Italic	Transliteration
А а	*А а*	A, a	Р р	*Р р*	R, r
Б б	*Б б*	B, b	С с	*С с*	S, s
В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
Е е	*Е е*	Ye, ye; E, e*	Х х	*Х х*	Kh, kh
Ж ж	*Ж ж*	Zh, zh	Ц ц	*Ц ц*	Ts, ts
З з	*З з*	Z, z	Ч ч	*Ч ч*	Ch, ch
И и	*И и*	I, i	Ш ш	*Ш ш*	Sh, sh
Й й	*Й й*	Y, y	Щ щ	*Щ щ*	Shch, shch
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М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

*ye initially, after vowels, and after ъ, ь; e elsewhere.
When written as ё in Russian, transliterate as yё or ё.

RUSSIAN AND ENGLISH TRIGONOMETRIC FUNCTIONS

Russian	English	Russian	English	Russian	English
sin	sin	sh	sinh	arc sh	$\sinh^{-1}$
cos	cos	ch	cosh	arc ch	$\cosh^{-1}$
tg	tan	th	tanh	arc th	$\tanh^{-1}$
ctg	cot	cth	coth	arc cth	$\coth^{-1}$
sec	sec	sch	sech	arc sch	sech^{-1}
cosec	csc	csch	csch	arc csch	csch^{-1}

Russian	English
rot	curl
lg	log

PROBLEMS OF DIFFRACTION AND PROPAGATION OF ELECTROMAGNETIC WAVES.

V. A. FoK.

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Preface.

New physical concepts are created not only in the process of generalizing the physical theories, but also by the return route: they can arise as a result of applying the approximation methods to a more precise physical theory. On the fundamental value of approximation methods in theoretical physics for us it was already necessary to write in connection with quantum mechanics [44]. Since this book is dedicated to electromagnetic theory, we will examine here examples of the emergence of new concepts in the theory of light/world (both in the electromagnetic theory and in the simpler wave theories of light/world preceded it).

So, the concept of ray/beam, and equally all geometric optic/optics can be derived from the wave theory of light/world as the idealizations, suitable in the extreme case of very small wavelength (in the region near the boundary of light/world and shadow these idealizations are already unsuitable). During the less complete idealization are considered the divergences from the geometric optic/optics, in other words, it is considered the diffraction, such as it is also the new physical concept (diffractive phenomena are

most clearly seen exactly near the boundary between the light/world and the shadow).

In the first investigations of diffractive phenomena the specific properties of the material of the diffracting body usually were not considered: the body was accepted as the absolutely black (absorbing). Laws of reflection which consider these properties, in the analysis of diffractive phenomena were not utilized. As the basis of the theory of diffraction was assumed/set Huygens-Fresnel's principle; on this basis light field near the boundary of light/world and shadow was described by means of Fresnel's integrals, and field near caustics - by means of Airy's integrals.

With the emergence, about hundred years ago, electromagnetic theory of light (applied also to the radio waves) it became necessary to anew formulate the theory of diffraction. As reliable theoretical basis for describing the diffraction phenomena serve the equations of Maxwell with the appropriate limiting conditions and the conditions for radiation/emission.

The contemporary asymptotic theory of diffraction can be defined as the approximation/approach to Maxwell's theory, suitable in the extreme case of small wavelengths and large radii of curvature of the diffracting bodies (we eliminate from the examination of case of

chisel edges).

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The asymptotic theory of diffraction introduced its inherent principles. Is established/installed the principle of local field into the region of penumbra on the surface of convex body, and also the generalization of this principle, which makes it possible to apply it, also, to the region, which adjoins the body. Further we have the approximate form of limiting conditions on the surface of the well conductive body (Leontovich); Leontovich's conditions are impedance type conditions and escape/ensue from the local character of field in the surface layer of conductive body. These principles allowed us to obtain explicit field expressions close to and on the very surface of the well conducting convex body of arbitrary form. (Our formulas, although completely general/common/total, were obtained by applying the principle of local field to the case of diffraction on the paraboloid of revolution; their direct conclusion/output from the 2 integral equations established/installed by us in chapter it was subsequently given by Kellon [28].).

Further principle is the introduction of different scales for the horizontal and vertical distances above the conductive body (above the surface of the Earth); this is allowed (Leontovich). The

latter takes the form of the quantum-mechanical equation of Schroedinger (or the equation of diffusion with the alleged diffusion coefficient), in whom the time is replaced to the horizontal coordinate. Parabolic equation makes it possible to introduce the concept of transverse diffusion (Malyuzhinets).

In the first part of this book is constructed the asymptotic theory of diffraction. The construction of theory is realized in two ways: first, on the basis of strict solutions of equations of Maxwell (these solutions are converted to the approximate form, which allows/assumes physical interpretation and suitable for the numerical calculations) and, in the second place, by the direct use/application of principles indicated above, in particular parabolic equation. In certain cases are applied both methods; this makes it possible to compare them between themselves and gives supplementary substantiation to "asymptotic" method. In the majority of the cases the field is examined only near the surface of the diffracting body, but some formulas of latter/last Chapter One part relate also to the large distances from the surface. The necessary tables are given in addition 3.

In the second part of the book is developed the theory of radiowave propagation. In first Chapter Two part is examined the propagation in homogeneous atmosphere; the object/subject of these

chapters pertains, thus, to the diffraction in the true sense, and they are the straight/direct continuation of the first part. In further chapters is examined the heterogeneous (laminar) atmosphere with the refractive index, which depends only on the height/altitude.

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It is here appropriate to note that the case of stratified atmosphere gives another series/row of examples of the new physical concepts, which appear as a result of applying the approximation methods. Such concepts are: the concept of the given refractive index (combining the effect of refraction with the effect of the curvature of the earth/ground), the concept of radio duct and finally the concept of the horizons/levels, determining the range of propagation in the case of superrefraction. Let us note that with the concept of radio duct are connected two different concepts, namely: the seized waves (oscillation) and the waves (rays/beams), reflected from the boundaries of waveguide. These two concepts are supplementary, in the sense that about the reflection in the sense of geometric optic/optics it is possible to speak only if excited not one, but there are many oscillations.

The latter/last chapter, dedicated to the phenomenon of coastal refraction, is at the same time an example of use/application to the

tasks of the diffraction of the integral equations of the specific form (namely: equations with the semi-infinite limits and with the nucleus, depending on the absolute value of a difference in the arguments). Since integral equations of this form are encountered both in the task about the radiation/emission of waveguide with the open end/lead and also in other tasks of mathematical physics, we give in addition 1 complete mathematical theory of such integral equations. In addition 2 is given the survey/coverage of properties and uses/applications of Airy's functions, which play very large role in the asymptotic theory of diffraction, and are also given the four-place tables of these functions.

The target of this book is, in the first place, the presentation of the general theory, developed in the investigations of the author, and only in the second turn - the analysis of numerical results. Nevertheless, when the demonstrative quantitative interpretation of general formulas is difficult, we give also numerical results in the form of graphs/curves and tables.

For facilitating the use of the book we premise to each chapter and each addition short abstract; this will allow the reader with the fugitive survey of the book to already obtain representation about her content.

Since this book represents the collector/collection of the original work of the author in their initial form (with the insignificant changes), in it are unavoidable the repetitions. We think, however, that this book with a sufficient clarity shows and logical the development of the ideas, which compose the object/subject of the works of the author on the theory of diffraction and propagation of electromagnetic waves.

Some works, connected with this collector/collection, are written by me with the co-authors whose names are indicated in the appropriate chapters. Me it would like to express here to all my co-authors her deep appreciation for the collaboration.

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The first edition of this book left in 1965 in the English language as the first volume of the series of monographs on the electromagnetic waves, the international editorial board published on the resolution. (V.A.Fock, Electromagnetic Diffraction and Propagation Problems, Pergamon Press, London, 1965, Vol. I of the International Series of Monographs in Electromagnetic Waves).

This Russian edition differs from English in terms of the inclusion/connection of chapter 18, dedicated to coastal refraction,

and addition 1, dedicated to the theory of integral equations. It represents, thus, the expanded (in comparison with English) meeting of the work of the author on this question. Although all, or almost all, these works were already printed in Russian in the form of various articles, to use them is sometimes difficult, since these articles were scattered on the different journals, and the time of their publishing covers the almost twenty-year period (from 1944 through 1963). Meanwhile how we can judge, our results did not cease to be urgent/actual. Therefore we hope that this edition of our works on the theory of diffraction and propagation of electromagnetic waves answers the actually/really ripened necessity.

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Chapter 13.

PROPAGATION OF DIRECT WAVE AROUND THE EARTH WITH CONSIDERATION OF DIFFRACTION AND REFRACTION. [1].

FOOTNOTE [1]. Fok, 1948. ENDFOOTNOTE.

Is derived/concluded approximate solution of the equations of Maxwell for the vector of hertz of vertical dipole, which considers both the diffraction, and refraction. This solution befits under the very general/common/total assumptions about the course of the refractive index of air in the dependence on the height/altitude. In some practically important cases the solution can be expressed through the functions, introduced for uniform atmosphere. Here the significant role plays the concept of an equivalent radius of the Earth. This concept, initially introduced on the basis of considerations about the curvature of ray/beam, can be substantiated and generalized. It proves to be applicable also in the shadow zone and penumbra where the geometric optic/optics is not applicable.

Introduction.

Under the assumption of the uniformity of the earth's surface the radiowave propagation around the Earth is caused, in essence, by the following three facts: diffraction around the convexity of the Earth, by refraction in the lower layers of the atmosphere and by reflection from the ionosphere. At the small distances, the order of hundred or several hundred kilometers, the reflection from the ionosphere role does not play. However, at the distances of the order of thousand or several thousand kilometers the reflection from the ionosphere begins to play the significant role, since to the straight/direct (diffractive) wave, which did not undergo reflection, begin to be superimposed the waves reflected, which can have considerably greater intensity than direct wave.

However, at these large distances it is possible, under the known conditions, to isolate direct wave and to observe it separately. Its study is of great practical interest for interference methods of the determination of distances.

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Therefore the construction of theory, which would give amplitude and phase of direct wave up to the largest distances, represents task very important for the practice.

The theory of direct wave must consider both the diffraction and refraction. Meanwhile in view of the complexity of task in the majority of theoretical studies atmospheric refraction either was not considered completely or it was considered very roughly, by the methods of geometric optic/optics. Very important in this question concept about equivalent radius of the Earth did not obtain a sufficient theoretical substantiation. It was introduced on the basis of considerations about the curvature of ray/beam, while in the region of penumbra, and those more in the shadow zone, the concept of ray/beam generally becomes meaningless. In connection with this were not explained those conditions, with executing of which the replacement of a radius of the Earth by an equivalent radius is legal.

In the present work we will give approximate solution of the Maxwell equations for the vector of hertz, that considers as diffraction. so refraction. This solution befits under the very general/common/total assumptions about the course of the refractive index of air in the dependence on the height/altitude.

In some practically important cases this solution can be expressed through the functions, introduced by us in our solution of the problem about the radiowave propagation in homogeneous atmosphere. These functions partly are already tabulated; when there

are tables, the calculation of field taking into account refraction does not compose large labor/work. Incidentally we will give the substantiation of the concept of an equivalent radius of the Earth, will show that this concept is applicable in the shadow zone and penumbra (where the geometric optic/optics is not applied), and let us explain those conditions when the use of it is lawful.

1. Differential equations and limiting conditions of task.

Let us designate through r, ϑ, φ spherical coordinates with the beginning in the center of terrestial globe and with the polar axis, passing through the radiating dipole. Dipole we will assume arranged/located on the earth's surface and will study the field of in the air. We designate the Earth's radius by a. The dielectric constant of air we will consider function from height/altitude h=r-a above the earths surface.

$$\varepsilon = \varepsilon(h), \quad h = r - a. \tag{1.01}$$

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As in the case of homogeneous atmosphere, field component in the air can be expressed through the function of hertz U. We have:

$$E_r = \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\,\frac{\partial U}{\partial\vartheta}\right), \quad (1.02)$$

$$E_\vartheta = -\frac{1}{\varepsilon r}\frac{\partial}{\partial r}\left(\varepsilon r\,\frac{\partial U}{\partial\vartheta}\right), \quad (1.03)$$

$$H_\varphi = -ik_0\varepsilon\,\frac{\partial U}{\partial\vartheta}, \quad (1.04)$$

whereas remaining field component are equal to zero. The dependence of field component on the time we assume in the form $e^{-i\omega t}$, where.

$$\frac{\omega}{c} = k_0 = \frac{2\pi}{\lambda_0}. \quad (1.05)$$

Here λ_0 - wavelength in the void (in its our task it is necessary to differ from wavelength in the air).

The value of the dielectric constant of air on the earth's surface we will designate through $\varepsilon_0 = \varepsilon(0)$ and let us assume.

$$k = \frac{2\pi}{\lambda} = k_0\sqrt{\varepsilon_0}. \quad (1.06)$$

The field, expressed by formulas (1.02)-(1.04), will satisfy the equations of Maxwell, if function U satisfies equation

$$\frac{\partial}{\partial r}\left(\frac{1}{\varepsilon}\frac{\partial}{\partial r}(\varepsilon r U)\right) - \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\,\frac{\partial U}{\partial\vartheta}\right) - k_0^2\varepsilon r U = 0. \quad (1.07)$$

Let us introduce new function

$$U_1 = \varepsilon r\sqrt{\sin\vartheta}\,U. \quad (1.08)$$

This function must satisfy equation

$$\frac{\partial}{\partial r}\left(\frac{1}{\varepsilon}\frac{\partial U_1}{\partial r}\right) + \frac{1}{\varepsilon\cdot r^2}\left[\frac{\partial^2 U_1}{\partial\vartheta^2} + \left(\frac{1}{4} + \frac{1}{4\sin^2\vartheta}\right)U_1\right] + k_0^2 U_1 = 0. \quad (1.09)$$

Field must satisfy on the earth's surface Leontovich's conditions:

$$E_\vartheta = -\frac{1}{\sqrt{\eta}}H_\varphi, \quad (1.10)$$

where

$$\eta = \varepsilon_2 + i\frac{4\pi\sigma_2}{\omega} \qquad (1.11)$$

there is composite dielectric constant of soil.

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Leontovich's condition will be made, if function U_1 satisfies condition

$$\frac{\partial U_1}{\partial r} = -\frac{ik_0\varepsilon_0}{\sqrt{\eta}}U_1 \quad (\text{при } r = a). \qquad (1.12)$$

Let us isolate in function U_1 the rapidly changing factor, after placing

$$U_1 = e^{ika\vartheta}U_2 = e^{iks}U_2. \qquad (1.13)$$

where k has value (1.06) and $s=a\vartheta$ is the arc length of terrestial globe from the point where is located dipole, to the point, above which is calculated the field.

For function U_2 is obtained the equation

$$\frac{\partial^2 U_2}{\partial r^2} - 2i\frac{k}{a}\frac{\partial U_2}{\partial \vartheta} - k^2\left(\frac{\varepsilon}{\varepsilon_0} - \frac{a^2}{r^2}\right)U_2 =$$

$$= \frac{\varepsilon'}{\varepsilon}\frac{\partial U_2}{\partial r} - \frac{1}{r^2}\left[\frac{\partial^2 U_2}{\partial \vartheta^2} + \left(\frac{1}{4} + \frac{1}{4\sin^2\vartheta}\right)U_2\right]. \qquad (1.14)$$

Equation (1.14) is written so that in its left side stand the dominant terms, and in the right - correcting, which, as we will show, can be replaced with zero.

During the estimation of the order of magnitude of derivatives we can use the results, obtained for the case of homogeneous atmosphere. If we introduce the "high parameter"

$$m=\sqrt[3]{\frac{ka}{2}}, \qquad (1.15)$$

then it will be

$$\frac{\partial U_2}{\partial r}=0\left(\frac{k}{m}U_2\right), \quad \frac{\partial U_2}{\partial \theta}=0\left(\frac{ka}{m^2}U_2\right), \qquad (1.16)$$

where symbol 0(x) indicates, as it is accepted, the "value of order x".

On the other hand, if we exclude from the examination the ionosphere (in which ε it can turn into zero), then the gradient of logarithm there will be the order of the curvature of the earth's surface, so that

$$\frac{\varepsilon'}{\varepsilon}=0\left(\frac{1}{a}\right). \qquad (1.17)$$

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Hence it is apparent that the individual members of left side (1.14) will be by order of value not less $\frac{k^2}{m^2}U_2$, whereas in the right side the members, who contain derivatives, will be on the order of However, concerning the term, which contains $\sin^2\theta$ in the denominator, then under the condition $\frac{k^2}{m^4}U_2$.

$$ks \gg m \qquad (1.18)$$

it will be also small. Thus, throwing/rejecting the values of order

$1/m^2$ in comparison with unity, we can right side (1.14) replace with zero, after which we will obtain

$$\frac{\partial^2 U_2}{\partial r^2} + 2i\frac{k}{a}\frac{\partial U_2}{\partial \vartheta} + k^2\left(\frac{\varepsilon}{\varepsilon_0} - \frac{a^2}{r^2}\right)U_2 = 0. \qquad (1.19)$$

This is a parabolic equation of our task, the reminiscent of in form equation of Schroedinger of quantum mechanics.

In the obtained equation we can make further simplifications, after using the approximate equality

$$1 - \frac{a^2}{r^2} = 2\frac{h}{a}. \qquad (1.20)$$

After introducing, furthermore, instead of the angle ϑ arc length $s=a\vartheta$ and by considering s and h as the independent variables, we will obtain.

$$\frac{\partial^2 U_2}{\partial h^2} + 2ik\frac{\partial U_2}{\partial s} + k^2\left(\frac{\varepsilon - \varepsilon_0}{\varepsilon_0} - \frac{2h}{a}\right)U_2 = 0. \qquad (1.21)$$

Limiting condition for U_2 on the earth's surface there will be the same as for U_1, namely.

$$\frac{\partial U_2}{\partial h} = -ik\sqrt{\frac{\varepsilon_0}{\eta}}\,U_2 \quad (\text{при } h = 0). \qquad (1.22)$$

Condition at infinity ($h\to\infty$) can be obtained from the examination of the phase of the function of hertz. Let us assume

$$U = |U|e^{i\Phi} \text{ и } U_2 = |U_2|e^{i(\Phi - ks)}. \qquad (1.23)$$

Since we examine the wave, which goes om source, then phase Φ must grow/rise with an increase in altitude h. Hence we obtain the condition

$$\frac{\partial \Phi}{\partial h} > 0, \qquad (1.24)$$

which must be made at least with sufficiently large h.

Furthermore, the function of hertz U, and also function U_2 must remain final and continuous in entire space, with exception of the region, which adjoins the source.

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For the single-valued determination of the solution of equation (1.21) to us it remains to formulate the condition, equivalent to the volume which must satisfy function U_2 in the region, close to the source. First of all, it is obvious that in directly the nearness to the source is disrupted inequality (1.18) and equation itself (1.21) becomes wrong. Therefore condition in the region, close to the source, must be replaced by equivalent condition in the "wave zone". We can take, for example, the region where is applicable "reflecting formula", and obtain the desired condition, after requiring so that the unknown solution in this region would convert/transfer into it.

Reflecting formula takes the form

$$U = \frac{e^{ikR}}{R}(1 + f), \qquad (1.25)$$

where f - Fresnel's coefficient.

Since we use the limiting conditions of Leontovich (1.10), then

we thereby assume $\eta' \gg 1$. If, furthermore, we will consider that h<<s, i.e. to examine small inclinations of ray/beam to the earth's surface, then we can assume.

$$R = s - \frac{h^2}{2s}, \quad f = \frac{h\sqrt{\eta} - s}{h\sqrt{\eta} - s}. \tag{1.26}$$

Substituting these expressions in (1.25), we come to the conclusion that in that region where is applicable "reflecting formula", the function

$$U_2 = e^{-iks} \varepsilon r \sqrt{\sin\vartheta}\, U \tag{1.27}$$

must be reduced to the form

$$U_2 = \frac{t_0\sqrt{\varepsilon}}{\sqrt{s}} \frac{2h\sqrt{\eta}}{h\sqrt{\eta} - s} e^{ik\frac{h^2}{2s}}. \tag{1.28}$$

Mathematically this condition is equivalent to requirement so that with s->0 and h>0 the function U_2 would have a special feature/peculiarity, characterized by the formula

$$\lim_{s\to 0} \left(u_2 - \frac{2t_0\sqrt{\varepsilon}}{\sqrt{s}} e^{ik\frac{h^2}{2s}} \right) = 0. \tag{1.29}$$

The more detailed substantiation of condition (1.29) is given in Chapter 11.

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Let us note that instead of conditions (1.29) or (1.28) we could supply even the more rigorous condition, after requiring so that in that region where the effect of the curvature of the earth's surface

and heterogeneity of the atmosphere no longer is manifested and where is applicable Weyl-van der Pol's formula [1], our solution would convert/transfer into the decision of Weyl-van der Pol.

FOOTNOTE [1]. The region of the applicability of Weyl-van der Pol's formula is in detail investigated in work [22] and in Chapters 10 and 11. ENDFOOTNOTE.

2. Transition to are dimensionless values.

The differential equation derived by us for function U_2 takes the form.

$$\frac{\partial^2 U_2}{\partial h^2} + 2ik\frac{\partial U_2}{\partial s} + k^2\left(\frac{\varepsilon - \varepsilon_0}{\varepsilon_0} + \frac{2h}{a}\right) U_2 = 0. \qquad (2.01)$$

Let us examine coefficient with U_2 in this equation. After designating through ε_0' the value of the gradient of dielectric constant on the earth's surface, we can isolate in the expression for linear term and write coefficient with U_2 in the form

$$k^2\left(\frac{\varepsilon - \varepsilon_0}{\varepsilon_0} + \frac{2h}{a}\right) = k^2\left[\frac{\varepsilon - \varepsilon_0 - \varepsilon_0' h}{\varepsilon_0} + \left(\frac{2}{a} + \frac{\varepsilon_0'}{\varepsilon_0}\right) h\right]. \qquad (2.02)$$

Let us assume now

$$\frac{1}{a^*} = \frac{1}{a} + \frac{\varepsilon'}{2\varepsilon_0}. \qquad (2.03)$$

Value (2.03) is a difference between the curvature of the earth's surface and the curvature of ray/beam, and value a* is conventionally designated as an equivalent radius of the Earth. After accepting designation (2.03), we can formula (2.02) write in the form

$$k^2\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon_0'}+\frac{2h}{a}\right)=\frac{2k^2}{a^*}h(1-g), \qquad (2.04)$$

where

$$g=\frac{a^*}{2\varepsilon_0'}\left(\frac{\varepsilon-\varepsilon_0}{h}-\varepsilon_0'\right). \qquad (2.05)$$

As can be seen from (2.05), value g is the difference expressed in the arbitrary (deprived of dimensionality) units between the average/mean in the height/altitude value of the gradient of the dielectric constant of air (undertaken in the section from the earth's surface to the given height/altitude) and its value on the earth's surface. In the standard atmosphere value g is positive, whereas in the case of temperature inversion it can become negative and, only beginning from certain height/altitude, it again becomes positive. In terms of the absolute value value g usually is not more than two-three tenths.

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With $h\to\infty$ the theoretical value g exists $\frac{a^*-a}{a}$, while with h=0 there will be g=0. With the standard atmosphere value g is changed very slowly, and in the case of inversion its change occurs considerably more rapid.

Substituting expression (2.04) in differential equation (2.01),

we will obtain

$$\frac{\partial^2 U_2}{\partial h^2} - 2ik\frac{\partial U_2}{\partial s} + \frac{2k^2}{a^*} h(1+g) U_2 = 0. \qquad (2.06)$$

For the investigation of equation (2.06) it is convenient to switch over from h and s to dimensionless quantities. For this purpose let us introduce the vertical and horizontal scales

$$h_1 = \sqrt[3]{\frac{a^{*}}{2k^2}}, \quad s_1 = \sqrt[3]{\frac{2a^{*2}}{k}} \qquad (2.07)$$

and let us assume

$$\frac{h}{h_1} = y, \quad \frac{s}{s_1} = x. \qquad (2.08)$$

In order to simplify condition (1.29), we will switch over also to the new dimensionless function W_1, after assuming

$$U_2 = \frac{\varepsilon_0\sqrt{a}}{\sqrt{s_1}} W_1. \qquad (2.09)$$

Furthermore, let us assume

$$q = ikh_1\sqrt{\frac{\varepsilon_c}{\eta}} = i\sqrt[3]{\frac{ka^*}{2}}\sqrt{\frac{\varepsilon_c}{\eta}}. \qquad (2.10)$$

In the new designations differential equation, limiting condition and condition, which is determining special feature/peculiarity, will be written:

$$\frac{\partial^2 W_1}{\partial y^2} + i\frac{\partial W_1}{\partial x} - y(1-g)W_1 = 0, \qquad (2.11)$$

$$\frac{\partial W_1}{\partial y} - qW_1 = 0 \quad (\overset{(1)}{\text{при}}\ y = 0). \qquad (2.12)$$

$$\lim_{x\to 0}\left(W_1 - \frac{2}{\sqrt{x}}e^{i\frac{y^2}{4x}}\right) = 0 \quad (y > 0). \qquad (2.13)$$

Key: (1). with.

Furthermore, remains valid condition for phase $\Phi = ks + \arg W_1$, namely:

$$\frac{\partial\Phi}{\partial y} > 0 \quad (\overset{(1)}{\text{при}}\ y \gg 1). \qquad (2.14)$$

Key: (1). with.

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The entering equation (2.11) value g was defined above [formula (2.05)] as function from height/altitude h. Let us designate through h_0 certain height/altitude, which characterizes the speed of a change in the gradient of the dielectric constant of air, for example that height/altitude for elongation/extent of which the gradient is changed in e=2.718... once. (For standard atmosphere h_0=7400 m; in other cases it is possible to indicate only the order of magnitude h_0, which to us only is needed.) The value of g we can consider as the function of ratio h/h_0:

$$g = g\left(\frac{h}{h_0}\right), \quad g(0) = 0, \qquad (2.15)$$

considering that the derivative of this function of its argument there will be the order of one. Upon transfer to dimensionless quantities (2.08) we had to consider g as function from y. So $h=h_1y$, we will have

$$g = g(\beta y), \qquad (2.16)$$

where

$$\beta = \frac{h_1}{h_0} = \frac{1}{h_0} \sqrt[3]{\frac{a^*}{2k^2}}. \qquad (2.17)$$

Subsequently we will consider the parameter β low value. In order to rate/estimate its order, let us assume h_0=7400 m (standard atmosphere) and let us replace the equivalent radius a* with the geometric radius a. Then for λ=1 m, λ=10 m, λ=100 m, λ=1000 m is obtained respectively β=0.006, β=0.027, β=0.13, β=0.58. In the case of inversion value h_0 will be considerably less, and the parameter β will prove to be small only for the respectively smaller wavelengths.

3. Solution of equations.

If we in the equation

$$\frac{\partial^2 W_1}{\partial y^2} - i \frac{\partial W_1}{\partial x} + y [1 - g(\beta y)] W_1 = 0 \qquad (3.01)$$

will count β=0, then, since g (0)=0, function g identically will become zero, and equation will be led to that such as was examined and solved [together with the limiting conditions (2.12) and (2.13)] in the preceding/previous chapters of this book, dedicated to the investigation of the case of homogeneous atmosphere. It is important, however, to note that the condition β=0 corresponds not to assumption about the uniformity of the atmosphere, but to more general assumption about the constancy of the gradient of dielectric

constant. In this more general case of formula are obtained the same as in the case of homogeneous atmosphere, only in the expressions for x, y and q enters instead of a radius of Earth a the equivalent radius a*.

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Thus, the smallness of value β characterizes the degree of accuracy with which it is possible to use (for the finite values of y) the concept of an equivalent radius.

Solution for β=0 have obtained we in the form of the integral, which contains composite Airy's function. The latter is that solution of the differential equation

$$w''(t) = tw(t), \tag{3.02}$$

which has with large negative t the asymptotic expression

$$w(t) = e^{i\frac{\pi}{4}}(-t)^{-\frac{1}{4}}e^{i\frac{2}{3}(-t)^{3/2}}. \tag{3.03}$$

Solution for $g(\beta y)$=0 takes the form

$$W_1 = e^{-i\frac{\pi}{4}}\frac{1}{\sqrt{\pi}}\int\limits_C e^{ixt}\frac{w(t-y)}{w'(t)-qw(t)}\,dt, \tag{3.04}$$

where duct/contour C goes from t=i∞ to t=0 and from t=0 to $t = \infty\cdot e^{i\alpha}\left(0 < \alpha < \frac{\pi}{3}\right)$, covering all roots of the denominator of integrand. (This duct/contour it is possible it goes without saying

to replace with some other equivalent contour.) this solution coincides with the fact which was obtained earlier in Chapter 10 and in book [22].

Let us attempt analogously to find the solution of our equations for the general case $\beta \neq 0$. In this case we will not first assume β small and only then for the purpose of simplification in the finished solution we will use assumption about the smallness β.

Equation (3.01) allows/assumes separation of variables the particular solution of equation (3.01), which has the form of the product of function from x for the function from y and containing the arbitrary parameter t, will be written

$$W_1 = e^{ixt} f(y, t), \qquad (3.05)$$

where f(y, t) satisfies the equation

$$\frac{d^2 f}{dy^2} + [y - t + yg(\beta y)] f = 0. \qquad (3.06)$$

From the theory of differential equations it is known that if the initial values (i.e. the value with y=0) of function f and its derivative on y are the integral functions of parameter t, then integral of equation (3.06) will be whole transcendental function from t.

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We will understand by f(y, t) that integral of equation (3.06), which is whole transcendental function from t and allows/assumes at the high values of difference y-t (or its real part) the asymptotic representation

$$f(y, t) = \frac{Ce^{i\frac{\pi}{4}}}{\sqrt[4]{y-t+yg(\beta y)}} \exp\left[i\int_{\tau}^{y}\sqrt{u-t+ug(\beta u)}\,du\right]. \quad (3.07)$$

Lower limit τ in the integral, which stands in the index, can be undertaken arbitrarily. Factor C can be function from parameter t. Phase factor $e^{i\frac{\pi}{4}}$ is added so that with g=0 and τ=t expression (3.07) would convert/transfer into the asymptotic expression for the function

$$f(y, t) = Cw(t-y). \quad (3.08)$$

Expression (3.07) is undertaken in accordance with requirement $\frac{\partial\Phi}{\partial y} > 0$, by that superimposed for the phase.

Understanding by f(y, t) the recently definite integral of equation (3.06), let us examine the expression

$$W_1 = e^{i\frac{3\pi}{4}} \frac{1}{\sqrt{\pi}} \int_{\Gamma} e^{ixt} \frac{f(y, t)}{\left(\frac{\partial f}{\partial y} + qf\right)_{y=0}} dt, \quad (3.09)$$

where the duct/contour Γ takes the form, analogous to duct/contour C in integral (3.04).

Let us first of all note that the integrand in it is determined by the conditions superimposed above unambiguously, since remaining indefinite in (3.07) factor C in it it is shortened.

Further, integrand in (3.09) presents meromorphic complex variable function t: unique singular points in it are the roots of denominator.

The investigation of the roots of denominator in (3.09) is difficult to conduct with complete strictness. For this investigation it is necessary to know the behavior of function g(βy) at the composite values of y near the half-line arc y=π/3. However, on the basis of some not completely strict considerations which we do not here give, it is possible to expect that if function g(βy) in complex domain indicated will remain small (for example, $|g|<\frac{1}{2}$), then roots will be arranged/located in the manner that in the case of g=0, i.e., in the first quadrant of plane t near the half-line ars t=π/3. In any case this will be so at the low values of the parameter β.

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We should also know the behavior of function f(y, t) at the

positive values of t-y (and also in certain sector of plane t, which switches on positive real axis). The unknown asymptotic eqpression will be obtained by the analytical continuation of expression (3.07) through the third and fourth quadrants of plane t, since in the first are arranged/located roots f(y, t). It will take the form

$$f(y, t) = \frac{C}{\sqrt[4]{t - y - yg(\beta y)}} \exp\left[\int_{i}^{\tau} \sqrt{t - u - ug(\beta u)}\, du\right]. \quad (3.10)$$

If we assume here g=0 and to take τ=t, then this expression will be led as (3.07), to the asymptotic expression for function (3.08).

Knowing the location of roots and the behavior of integrand on both sides of the region where these roots are arranged/located, already it is possible to conduct in integral (3.09) duct/contour Γ. so that it would cover all roots of denominator and would become by both branches infinite. On the initial branch of duct/contour (going from infinity) there will be correctly asymptotic expression (3.07), while on the final branch (exiting to infinity) - expression (3.10). In this case the integral, undertaken on this duct/contour, will be that converging.

The preceding/previous reasonings had as a goal to show that expression (3.09) for function W_1 makes thinned mathematical sense.

Let us show now that it satisfies all conditions presented.

It is first of all clear that it satisfies differential equation (3.01) because it is satisfied by the integrand. Further it satisfies limiting condition (2.12):

$$\frac{\partial W_1}{\partial y} + qW_1 = 0 \qquad (\text{при } y = 0). \qquad (3.11)$$

Key: (1). with.

Actually/really, producing in (3.09) the differentiation under the integral sign and assuming/setting then y=0, we will see, that the numerator of fraction will be shortened with the denominator and integrand will be holomorphic, in consequence of which the integral will become zero. Then, integral will be convergent and, therefore, final at all positive values of x and y. It is not difficult to also check that it will satisfy condition for phase $\left(\frac{\partial \Phi}{\partial y} > 0\right)$.

To us it remains to check, does have expression (3.09) required by condition (2.13) special feature/peculiarity near x=0, or which is equivalent to this, to be convinced of the fact that at the small distances from the source it gives the formulas of Weyl-van der Pol or reflecting formula.

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With the aid of asymptotic expressions (3.07) and (3.10) for f(y, t) it is possible to show that if x and y are small, and ratio y/x is great, then the main section of integration will lie/rest at the region of the high negative values t. (Initial contour Γ can be deformed so that it would pass through this region.). Using expression (3.07), we will obtain for large negative t:

$$\frac{f(y,\,t)}{f(0,\,t)} = \sqrt[4]{\frac{-t}{y-t+yg(\beta y)}}\exp\left[i\int_0^y \sqrt{u-t+ug}\,du\right]. \quad (3.12)$$

Hence

$$\frac{1}{f}\frac{\partial f}{\partial y} = i\sqrt{y-t+yg(\beta y)} \quad (3.13)$$

and

$$\left(\frac{1}{f}\frac{\partial f}{\partial y}+q\right)_{y=0} = i\sqrt{-t}+q. \quad (3.14)$$

But, when y is small, terms yg(βy) are small in comparison with y, and we can instead of (3.12) write

$$\frac{f(y,\,t)}{f(0,\,t)} = \sqrt[4]{\frac{-t}{y-t}}\exp\left[i\int_0^y \sqrt{u-t}\,du\right]. \quad (3.15)$$

Let us note now that the same asymptotic expressions will be obtained for this region, if we instead of f(y, t) substitute

$$f(y,\,t) = w(t-y). \quad (3.16)$$

But after this substitution integral (3.09) will become (3.04), and

this latter gives for small x, y the formula of Weyl - van der Polya, reflecting formula and limiting condition (2.13).

We can check this and it is direct. By introducing the variable/alternating of integration $p = \sqrt{-t}$ and disregarding values y and y^2 in comparison with p, we will obtain

$$\frac{f(y, -p^2)}{f(0, -p^2)} = e^{iyp} \qquad (3.17)$$

and

$$\left(\frac{1}{f}\frac{\partial f}{\partial y} + q\right)_{y=0} = ip + q. \qquad (3.18)$$

The substitution of these values into integral (3.09) gives

$$W_1 = e^{-i\frac{3\pi}{4}} \frac{2}{\sqrt{\pi}} \int\limits_{\Gamma_2} e^{-ixp^2 + iyp} \frac{p\,dp}{p - iq}, \qquad (3.19)$$

where duct/contour Γ_2 · it intersects positive real axis in plane p from below upward (near point p=y/2x).

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If we will calculate integral (3.19) without the neglects, then will be obtained the formula of Weyl - van der Polya. If we will calculate it according to the method of steady state, obtain reflecting formula. If we disregard/neglect value $|q|$ in comparison with y/2x will be obtained the expression, which is turned into zero left sides (2.13) even to the transition to the limit.

Thereby it is proved that expression (3.19) for W_1 presents the

unknown solution of our problem.

4. Investigation of the solution for the zone of straight/direct visibility.

Instead of function W_1 to more conveniently examine another function, which differs from W_1 in terms of factor $\sqrt{x}$. We will assume

$$V(x, y, q) = e^{i\frac{3\pi}{4}} \sqrt{\frac{x}{\pi}} \int_{\Gamma} e^{ixt} \frac{f(y, t)}{\left(\frac{\partial f}{\partial y} + qf\right)_0} dt. \qquad (4.01)$$

Recollecting connection/communication between functions U, U_1, U_2, W_1, given by formulas (1.08), (1.13), (2.09), and disregarding the difference between r and a and between ε and ε_0, when these values enter as the factor with U, we can write

$$U = \frac{e^{iks}}{\sqrt{as \sin \frac{s}{a}}} V(x, y, q), \qquad (4.02)$$

where s, as before is the horizontal distance, counted on the arc along the earth's surface, and x, y, q are connected with s, h, η with the relationships/ratios

$$x = \frac{s}{s_1}, \quad y = \frac{h}{h_1}, \quad q = i\sqrt[3]{\frac{ka^*}{2}}\sqrt{\frac{\varepsilon_0}{\eta}}, \qquad (4.03)$$

moreover

$$s_1 = \sqrt[3]{\frac{2a^{*2}}{k}}, \quad h_1 = \sqrt[3]{\frac{a^*}{2k^2}}. \qquad (4.04)$$

If s is small in comparison with a radius of terrestial globe, then instead of sin s/a it is possible to write simply s/a (as usually and they write).

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However, since our formulas remain valid up to the very large distances where the difference between sine and another becomes noticeable, we leave in (4.02) under root of sin s/a.

Function V(x, y, q) can be named/called attenuation factor; when is possible to count g=0 and to use the concept of an equivalent radius, formula (4.01) for V converts/transfers into the formula

$$V(x,\ y,\ q)=e^{-i\frac{\pi}{4}}\sqrt{\frac{x}{\pi}}\int_{\Gamma} e^{ixt}\,\frac{w(t-y)}{w'(t)-q\,w(t)}\,dt. \qquad (4.05)$$

that derived by us earlier for the case of homogeneous atmosphere [formula (6.02) of chapter 10].

The function (4.05) it is in detail investigated in chapter 10 and in our work [22] and it is partly tabulated (for q=0). Investigation will flow/occur/last in many respects in parallel to analogous investigation in chapter 10.

In the present paragraph we will examine the zone of straight/direct visibility.

In the zone of straight/direct visibility, it is not too close to the horizon, comes into force geometric optic/optics. If we will

use expression (3.12) and let us introduce the variable/alternating of integration p, we will obtain for the V integral of the form

$$V = e^{-i\frac{3\pi}{4}} \frac{2}{\sqrt{\pi}} \sqrt{x} \int e^{i\omega} \sqrt[4]{\frac{p^2}{y - p^2 + yg(\beta y)}} \frac{p\,dp}{p - iq}, \qquad (4.06)$$

where for the brevity is placed [1]

$$\omega = -xp^2 - \int_0^y \sqrt{u - p^2 + ug(\beta u)}\,du. \qquad (4.07)$$

FOOTNOTE [1]. Phase ω should not be mixed with the angular frequency, designated in paragraph 1 by the same letter. ENDFOOTNOTE.

Calculating integral according to the method of steady states, we find from equation $\frac{\partial\omega}{\partial p} = 0$ or

$$x = \frac{1}{2} \int_0^y \frac{du}{\sqrt{u - p^2 + ug(\beta u)}} \qquad (4.08)$$

the extremum of phase and after some linings/calculations come to the expression

$$V = e^{i\omega} \frac{2p}{p - iq} \sqrt{2x \frac{\partial p}{\partial y}}. \qquad (4.09)$$

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In this formula under p is understood the function from x and y, determined from equality (4.08). With g=0, and also with small x and y

$$p = \frac{y - x^2}{2x}, \qquad (4.10)$$

and the expression under the sign of square root in (4.09) will become unity. Formula (4.09) is valid when value p is great and positive.

Our formulas allow/assume simple interpretation from the point of view of geometric optic/optics. Actually/really, the complete phase

$$\Phi = ks - \omega \qquad (4.11)$$

of function U presents the solution of equation for the eikonal

$$\left(1 + \frac{n}{a}\right)^2 \left(\frac{\partial\Phi}{\partial n}\right)^2 - \left(\frac{\partial\Phi}{\partial s}\right)^2 = k^2\left(1 - \frac{h}{a}\right)^2 \frac{\varepsilon}{\varepsilon_0}, \qquad (4.12)$$

which after neglect of low values reduces to the following equation for ω:

$$\left(\frac{\partial\omega}{\partial n}\right)^2 - 2k\frac{\partial\omega}{\partial s} = \frac{2k^2}{a} h(1 - g). \qquad (4.13)$$

Here to the right stands value (2.04). After transition to to the variable/alternating x, y we obtain from (4.13)

$$\left(\frac{\partial\omega}{\partial y}\right)^2 + \frac{\partial\omega}{\partial x} = y + yg(\beta y). \qquad (4.14)$$

Relationship/ratio (4.08) is an equation of the trajectory of the ray/beam, passing through the origin of coordinates, and value p - the parameter of this trajectory. The geometric value of parameter p exists

$$p = \sqrt[3]{\frac{ka^2}{2}}\cos\gamma, \qquad (4.15)$$

where γ - angle between the ray/beam and the vertical line near the source.

Complete phase Φ is optical path length of ray/beam, counted from the source to point x, y. Value 2p/p-iq is equal to

$$\frac{2p}{p - iq} = 1 - f. \qquad (4.16)$$

where f - Fresnel's coefficient.

Thus, when is applicable geometric optic/optics, our formulas convert/transfer into the formulas of geometric optic/optics.

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Formula (4.09) is applicable for final x and y when parameter p is positive and great. But if x and y are small, is necessary another condition

$$\frac{y^2}{4x} = \frac{kh^2}{2s} \gg 1. \qquad (4.17)$$

With the nonfulfillment of condition (4.17), in the case small x and y and large p, expression (4.06) remains valid but integral must be calculated otherwise, namely: it is necessary to replace ω by $-xp^2+yp$ and the root of the fourth degree to one, after which the integral will be reduced to form (3.19) (with factor $\sqrt{x}$) and it will give the formula of Weyl - van der Polya.

Let us note that if x and $\sqrt{y}$ are great, and parameter p is low in comparison with these values, then the equation of trajectory (4.08) can be approximately solved relative to p. We will have approximately

$$p = \frac{1}{2}\int_0^y \frac{du}{\sqrt{u - ug(\beta u)}} - x. \qquad (4.18)$$

Under the same conditions it is obtained

$$\omega = \omega_0(y) + \frac{1}{3}p^3, \qquad (4.19)$$

where

$$\omega_0(y) = \int_0^y \sqrt{u + ug(\beta u)}\,du, \qquad (4.20)$$

but symbol p it is necessary to understand as the abbreviation for value (4.18).

Equality p=0 gives the geometric boundary of shadow. If right side (4.18) becomes negative, then equation (4.08) does not have real solution relative to p, whereas function (4.18) [and also (4.10)] retains sense, also, in this case. This apparent disagreement is explained by the fact that right side (4.08) is not analytic function from p near p=0.

Expressions (4.18) and (4.19) will be encountered to us in the region of the penumbra where the geometric optic/optics is already not applied.

5. Investigation of solution for the region of penumbra (final x and y).

The region of penumbra is characterized by the fact that in it parameter p, determined by formula (4.10), is a positive or negative number of order of one.

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If x and y are final, we can compose for the V series/row, arranged/located on the deductions of integrand. We will have

$$V(x, y, q) = e^{i\frac{\pi}{4}} 2\sqrt{\pi x} \sum_{n=1}^{\infty} \frac{e^{ixt_n}}{D(t_n)} \frac{f(y, t_n)}{f(0, t_n)}, \qquad (5.01)$$

where we assumed

$$D(t) = -\frac{1}{f(0, t)} \left(\frac{\partial^2 f}{\partial y\, \partial t} - q \frac{\partial f}{\partial t} \right)_{y=0}, \qquad (5.02)$$

moreover pores t is understood the root of the equation

$$\left(\frac{\partial f}{\partial y} - qf \right)_{y=0} = 0. \qquad (5.03)$$

If β is not small, then calculation according to these formulas is very complicated. Therefore subsequently we will be restricted to the case of small ones β. In this case, however, we will not consider small product βy, but we will examine also the high values y (order $1/\beta$ more).

If β it is small, then during the calculation of the first roots of function (5.03) we can replace g(βy) by the linear function

$$g(\beta y) = g'(0)\beta y = \beta_0 y. \qquad (5.04)$$

The physical value of coefficient β_0 exists

$$\beta_0 = h_1 \left(\frac{dg}{dh}\right)_0 = -\frac{h_1 a^2 \varepsilon_0''}{4\varepsilon_0} \qquad (5.05)$$

where h_1 - height scale, and ε_0'' · ·. ue of the second derivative of on the height/altitude on the earth's surface.

With small β_0 and finite y and t as approximate solution of equation (3.06) it is possible to take the function

$$f(y, t) = w(t-y) -$$
$$-\frac{\beta_0}{15}\left[(3y+2t)\,w(t-y) - (3y^2+4yt+8t^2)\,w'(t-y)\right]. \quad (5.06)$$

Substituting this expression in (5.03), we will obtain for the unknown root the approximation

$$t_n = t_n^0 + \frac{\beta_0}{15}\left[8\,(t_n^0)^2 - \frac{3+4t_n^0 q}{t_n^0 - q^2}\right], \qquad (5.07)$$

where t_n^0 - root of the equation

$$w'(t_n^0) - qw(t_n^0) = 0, \qquad (5.08)$$

in detail investigated in chapter 10 and in work [22].

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For function D(t) is obtained the expression

$$D(t) = (t - q^2)\left(1 - \frac{4}{3}\beta_0 t\right) + \frac{2}{3}\beta_0 q. \qquad (5.09)$$

The entering formula (5.01) high-altitude factors

$$f_n(y) = \frac{f(y, t_n)}{f(0, t_n)} \qquad (5.10)$$

can be obtained by the numerical integration of the differential equation

$$\frac{d^2 f_n}{dy^2} + [y - t_n - yg(\beta y)]\,f_n = 0 \qquad (5.11)$$

under the initial conditions

$$f_n(0)=1 \text{ и } f_n'(0)=-q. \qquad (5.12)$$

Thus far y is certain (whereas x can be great), obtained in this method values V(x, y, q) they will, with small ones β, differ little from the values for β=0. More or less essential difference can be revealed only in exponential factors e^{ixt_n}, of those giving fading and additional phase. Therefore correction it suffices to introduce into these exponential factors.

But if for special accuracy it is not required, it is possible to disregard this correction and it is simple to consider that in the case in question the expression for V(x, y, q) coincides with that derived for the case of homogeneous atmosphere (with condition of the replacement of a geometric radius of the Earth by an equivalent radius). Then it is possible to use all formulas and tables, obtained for this case.

6. Investigation of solution for the region of penumbra (large x and y).

There is greatest practical interest in the case, whereas when x and y is very great, value

$$p=\frac{1}{2}\int_0^y \frac{du}{\sqrt{u+ug(\beta u)}}-x \qquad (6.01)$$

final. We already indicated that value p=0 corresponds to the boundary of straight/direct visibility, the positive values p corresponding to the zone of straight/direct visibility, and the negative values p - region beyond the horizon/level.

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In this case during the calculation of integral (4.01) for V(x, y, q) it is necessary to have in mind that the main section of integration will correspond to the finite values of t, whereas y will be great. Therefore it is necessary to compose for f(y, t) such analytical expression which would be correctly both with the very large ones and at the finite values of y-t. This proves to be possible under the condition of smallness β.

Actually/really, let us introduce value X, determined by the equality

$$\frac{2}{3}(-X)^{3/2} = \int_{\tau}^{y} \sqrt{u - t + ug(\beta u)}\, du \qquad (6.02)$$

or

$$\frac{2}{3}X^{3/2} = \int_{y}^{\tau} \sqrt{t - u - ug(\beta u)}\, du, \qquad (6.03)$$

where τ - root of the equation

$$\tau - t + \tau g(\beta\tau) = 0. \qquad (6.04)$$

With small β_0 and final y and t

$$X = t - y - \frac{\beta_0}{15}(3y^2 + 4ty - 8t^2). \qquad (6.05)$$

Then the function

$$f(y, t) = \sqrt{-\frac{dy}{dX}}\, w(X) \qquad (6.06)$$

will present the solution of equation (3.06) with error in the order β^2 with final y and t and order $\beta^{3/2}$ with large y and final t. With the aid of expression (6.05) it is not difficult to check that in expansion (6.06) according to degrees β_0 the members of order to β_0 inclusively coincide with (5.06). However, expression (6.06) is correct even when (with large y) expansion (5.06) is not applicable. If value X is very great and negative (which will be with large y), then expression (6.06) is led to the following:

$$f(y, t) = \frac{e^{i\frac{\pi}{4}}}{\sqrt[4]{y - t + yg(\beta y)}} \exp\left[i \int_{\tau}^{y} \sqrt{u - t - ug(\beta u)}\, du\right]. \qquad (6.07)$$

The latter coincides with (3.07), if we there assume C=1 and to take as τ the root of equation (6.04). Thus, by means of formula (6.06) we ascertained that one and the same solution of equation (3.06) has with final y expression (5.06). And with large y - expression (6.07).

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We can now use during the calculation of the integral

$$V(x, y, q) = e^{i\frac{3\pi}{4}} \sqrt{\frac{x}{\pi}} \int_{\gamma} e^{ixt} \frac{f(y, t)}{\left(\frac{\partial f}{\partial y} - qf\right)_0} dt \qquad (6.08)$$

both expressions (5.06) and (6.07) simultaneously, namely substitute in the numerator expression (6.07) and in the denominator - expression (5.06). In this case it is possible both expressions to somewhat simplify. Throwing/rejecting small corrections, we will write instead of (5.06) simply

$$f(y, t) = w(t - y). \qquad (6.09)$$

However, in the formula (6.07) in the factor before the exponential function let us disregard/neglect value t in comparison with y, and index is replaced by the approximation

$$\int_{t}^{y} \sqrt{u - t + ug(\beta u)}\, du =$$

$$= \int_{0}^{y} \sqrt{u + ug(\beta u)}\, du - \frac{1}{2} t \int_{0}^{y} \frac{du}{\sqrt{u + ug(\beta u)}}. \qquad (6.10)$$

Using designations (4.18) and (4.20), we can write

$$f(y, t) = \sqrt{2 \frac{\partial p}{\partial y}}\, e^{i\frac{\pi}{4}} e^{i\omega_0 y - it(x+p)}. \qquad (6.11)$$

In the final analysis we replace function f(y, t) in the denominator by Airy's function, while in the numerator - by exponential function.

Substituting (6.09) and (6.11) into integral (6.08), we will obtain

$$V(x, y, q) = e^{i\omega_0(y)} \sqrt{2x \frac{\partial p}{\partial y}} \frac{1}{\sqrt{\pi}} \int_{\Gamma} e^{-ipt} \frac{dt}{w'(t) - qw(t)}. \quad (6.12)$$

The remaining integral can be considered known function. In chapter 10 [formula (6.20) it is designated through

$$V_1(-p, q) = \frac{1}{\sqrt{\pi}} \int_{\Gamma} e^{-ipt} \frac{dt}{w'(t) - qw(t)} \quad (6.13)$$

and it is in detail investigated. For cases of p=0 and $q = \sqrt{i}$ integral V_1 is tabulated [1].

FOOTNOTE [1]. Tables for q=0 are given in addition 3. ENDFOOTNOTE.

formula (6.12) gives attenuation factor for the region, close to the horizon.

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It is interesting to compare this formula with formula (4.09), valid in the region of the applicability of geometric optic/optics. Using (4.19), let us write expression (4.09) in the form

$$V = e^{i\omega_0(y)} \sqrt{2x \frac{\partial p}{\partial y}} \frac{2p}{p - iq} e^{\frac{i}{3} p^3}. \quad (6.14)$$

But in chapter 10 it is shown that function (6.13) has with large positive p the asymptotic expression

$$V_1(-p, q) = \frac{2p}{p - iq} e^{\frac{i}{3} p^3} \quad (6.15)$$

[formula (6.24) of chapter 10]. Thus, our formula (6.12) converts/transfers in the zone of straight/direct visibility into the

formula of geometric optic/optics.

With negative p the expression for $V_1(-p, q)$ can be written in the form

$$V_1(-p, q) = i2\sqrt{\pi} \sum_{n=1}^{n=\infty} \frac{e^{-ipt_n}}{(t_n - q^2)\,w'(t_n)}. \qquad (6.16)$$

When $|p|$ is great ($p<0$), this series/row is reduced to the first term which gives the attenuation of wave in the shadow zone according to the exponential law.

Function $V_1(-p, q)$ was for the first time introduced in our works, dedicated to diffraction from the body of the arbitrary form (see chapters 1, 2 and 5). In these works was established/installed the principle of local field in the region of penumbra and it was shown that in this region the field is expressed as function $V_1(-p, q)$, which has universal character.

The comparison of formulas (6.12) and (6.14) makes it possible to in a certain sense say that the wave reaches the horizon/level with the amplitude and the phase, which corresponds to the laws of geometric optic/optics, and at the horizon/level it undergoes diffraction according to the law of local field in the region of penumbra.

This picture is in complete agreement with L. I. Mandelstam's ideas about the fact that during the propagation of electromagnetic waves along the earth's surface the properties of soil are essential not along the entire trajectory of ray/beam, but only in that region where is arranged/located the transmitter located on the earth/ground or receiver ("takeoff" or "landing" area/site).

If we accept this picture, then solution obtained in this paragraph can be applied also to that case when the properties of the earth's surface in the different sections are not identical, when in function $V_1(-p, q)$ the composite parameter q corresponds to the properties of soil in that region where the ray/beam concerns the earth/ground.

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Chapter 14.

Theory of radiowave propagation in an unhomogeneous atmosphere for the elevated source [1]. ENDFOOTNOTE.

FOOTNOTE [1]. Foch, 1950. ENDFOOTNOTE.

The theory developed in the preceding/previous chapters of radiowave propagation is generalized to the case of the elevated dipole in the heterogeneous (laminar) atmosphere, moreover is considered as normal refraction, so also superrefraction.

In the beginning of chapter are extracted fundamental equations and limiting conditions of task. Then is examined the approximate form of equations (parabolic equation of Leontovich) with the appropriate limiting conditions and the conditions, which are determining special feature/peculiarity. Further is drawn analogy between the formulated task and the unsteady task of quantum mechanics. After transition to the dimensionless quantities are studied the properties of the particular solutions of differential equation from which is constructed the general solution in the form

of contour integral and in the form of series/row. General theory is applied to the case of superrefraction and is examined a schematic example. In conclusion are derived the approximation formulas for determining the attenuation factors and high-altitude factors. These formulas are analogous to the semi-classical formulas of quantum mechanics.

Introduction.

The theory of radiowave propagation in the atmosphere with the dielectric constant, which depends on height/altitude, have developed we in chapter 13 for the case, when source is the vertical electric dipole, arranged/located on the earth's surface. On the other hand, the case of the elevated source (horizontal and vertical, electrical and magnetic dipoles) was examined by us in chapter 12 under the assumption of homogeneous atmosphere. At present to chapter is studied the "combined" case: the elevated dipole and unhomogeneous atmosphere.

The formulas, derived in chapter 13 for the general case of the arbitrary course of refractive index, were there in more detail developed under the assumption of the normal refraction when radiowave propagation has qualitatively the same character as in homogeneous atmosphere.

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The case of the superrefraction when the lower layer of the atmosphere acquires the character of waveguide, it is of independent interest and deserves special examination. In present chapter we examine this case in more detail. For its qualitative characteristic of the highly useful proves to be the analogy with the unsteady task of quantum mechanics about spreading wave packet in the assigned force field: this analogy, apparently, remained, until now, of that not noticed.

1. Fundamental equations and limiting conditions.

Let us designate through r, ϑ, φ spherical coordinates with the beginning in the center of terrestial globe and with the polar axis, passing through the radiating dipole. A radius of the Earth let us designate through a. Dipole we will assume finding on height/altitude $h'=b-a$ above the earth's surface, so that its coordinates will be $r=b$, $\vartheta=0$. The dielectric constant of air ε we will consider function from height/altitude $h=r-a$ above the earth's surface.

Field in the air can be expressed according to the known

formulas through Debye's potentials u, v.

We have:

$$\left.\begin{aligned} E_r &= \frac{1}{r}\Delta^* u, \\ E_\vartheta &= -\frac{1}{\varepsilon r}\frac{\partial^2(\varepsilon r u)}{\partial r\,\partial\vartheta} - \frac{i\omega}{c\sin\vartheta}\frac{\partial v}{\partial\varphi}, \\ E_\varphi &= -\frac{1}{\varepsilon r\sin\vartheta}\frac{\partial^2(\varepsilon r u)}{\partial r\,\partial\varphi} - \frac{i\omega}{c}\frac{\partial v}{\partial\vartheta}; \end{aligned}\right\} \qquad (1.01)$$

$$\left.\begin{aligned} H_r &= -\frac{1}{r}\Delta^* v, \\ H_\vartheta &= \frac{i\omega}{c}\frac{\varepsilon}{\sin\vartheta}\frac{\partial u}{\partial\varphi} - \frac{1}{r}\frac{\partial^2(rv)}{\partial r\,\partial\vartheta}, \\ H_\varphi &= -\frac{i\omega}{c}\varepsilon\frac{\partial u}{\partial\vartheta} - \frac{1}{r\sin\vartheta}\frac{\partial^2(rv)}{\partial r\,\partial\varphi}. \end{aligned}\right\} \qquad (1.02)$$

The same expressions are applicable for the field lower than earth's surface, if we by ε understand the composite dielectric constant of soil. Dependence on the time is assumed here in the form of factor $e^{-i\omega t}$.

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Symbol Δ^* indicates Laplace's operator on the sphere:

$$\Delta^* u = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\,\frac{\partial u}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2 u}{\partial\varphi^2}. \qquad (1.03)$$

The equations of Maxwell will be satisfied, if functions u and v satisfy the equations

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{\varepsilon}\frac{\partial(\varepsilon r u)}{\partial r}\right) + \frac{\Delta^* u}{r^2} + \frac{\omega^2}{c^2}\varepsilon u = 0 \qquad (1.04)$$

and

$$\frac{1}{r}\frac{\partial^2(rv)}{\partial r^2} + \frac{\Delta^* v}{r^2} + \frac{\omega^2}{c^2}\varepsilon v = 0. \qquad (1.05)$$

The continuity of the tangential components of field will be provided, if there will be also continuous values

$$\varepsilon r u, \quad \frac{1}{\varepsilon}\frac{\partial(\varepsilon r u)}{\partial r}, \quad r v, \quad \frac{\partial(r v)}{\partial r}. \qquad (1.06)$$

Hence, via known reasonings, is obtained the approximate form of limiting conditions (Leontovich's condition). If we will assume k=ω/c, let us designate the composite dielectric constant of soil through η. and designation ε let us retain for the dielectric constant of air, then we will have

$$\frac{\partial(\varepsilon r u)}{\partial r} = -ik\frac{\varepsilon}{\sqrt{\eta}}(\varepsilon r u) \quad (\overset{(1)}{\text{при}}\ r = a) \qquad (1.07)$$

Key: (1). with.

and

$$\frac{\partial(r v)}{\partial r} = -ik\sqrt{\eta}(r v) \quad (\overset{(1)}{\text{при}}\ r = a). \qquad (1.08)$$

Key: (1). with.

In further field for which u≠0, v=0, we will call vertically polarized, and the field for which u=0, v≠0, horizontally polarized. The field of vertical electric dipole remains, in this sense, vertically polarized in entire space. The field of vertical magnetic dipole (horizontal framework) possesses everywhere horizontal polarization. However, horizontal electric dipole excites the fields both form: as it is horizontal, so also vertically polarized. In the case of homogeneous atmosphere the vertically polarized field decreases with the increase of distance more slowly than horizontally

polarized (see Chapter 12). Therefore field from the horizontal electric dipole at the small distances from the source will possess predominantly horizontal polarization, and at large distances (in the region far beyond the horizon/level) polarization will be predominantly vertical.

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The vertically polarized field can be expressed through function U (function of hertz of vertical electric dipole), which possesses the following properties: function U satisfies the same differential equation (1.04) and the same limiting conditions (1.07)that also u, and has near the source a special feature/peculiarity of the form

$$U = \frac{e^{ikR}}{R} + U^*, \qquad (1.09)$$

where U* remains final. Values R and k have values

$$R = \sqrt{r^2 + b^2 - 2rb\cos\vartheta}, \quad k = \frac{\omega}{c}. \qquad (1.10)$$

Analogously, horizontally polarized field can be expressed through function W (function of hertz of vertical magnetic dipole), which satisfies the same differential equation (1.05) and the same limiting conditions (1.08)that also v, and has near the source a special feature/peculiarity of the form

$$W = \frac{e^{ikR}}{R} + W^*. \qquad (1.11)$$

where W* remains final.

Fields from the vertical and horizontal electrical and magnetic dipoles with moment/torque M are expressed as the specific above functions U and W.

For the vertical electric dipole we must assume

$$u = \frac{M}{b} U, \quad v = 0. \qquad (1.12)$$

For the vertical magnetic dipole (horizontal framework) we have:

$$u = 0, \quad v = \frac{M}{b} W. \qquad (1.13)$$

For the horizontal electric dipole, directed along the axis x, entering formulas (1.01) and (1.02) functions u and v are determined from the equations

and

$$\left.\begin{aligned} \Delta^* u &= -M \frac{\partial}{\partial \theta} \left(\frac{\partial U}{\partial b} + \frac{U}{b} \right) \cos \varphi \\ \Delta^* v &= -ikM \frac{\partial W}{\partial \theta} \sin \varphi, \end{aligned}\right\} \qquad (1.14)$$

where Δ* - operator of Laplace on sphere (1.03).

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Finally, for the horizontal magnetic dipole, directed along the axis x, we have

and

$$\left.\begin{aligned} \Delta^* u &= -ikM \frac{\partial U}{\partial \theta} \sin \varphi \\ \Delta^* v &= M \frac{\partial}{\partial \theta} \left(\frac{\partial W}{\partial b} + \frac{W}{b} \right) \cos \varphi. \end{aligned}\right\} \qquad (1.15)$$

Thus, into all four cases the study of field is reduced to the study

of functions U and W.

2. Approximate form of equations.

Converting/transferring to the approximate form of equations, let us designate through ε_1 the value of the dielectric constant of air near the source (practically it is possible to assume $\varepsilon_1 = 1$) and let us assume

$$s = a\vartheta, \tag{2.01}$$

so that s - horizontal distance between source and observation point, counted on the arc.

Let us introduce instead of U and W slowly changing functions U_2 and W_2, after assuming

$$U = \frac{\varepsilon_1 e^{iks}}{\varepsilon r \sqrt{\sin\vartheta}} U_2 \tag{2.02}$$

and

$$W = \frac{e^{iks}}{r\sqrt{\sin\vartheta}} W_2. \tag{2.03}$$

As shown in chapter 13, after neglect of low values the equation for U_2 takes the form

$$\frac{\partial^2 U_2}{\partial h^2} - 2ik\frac{\partial U_2}{\partial s} + k^2\left(\varepsilon - 1 - \frac{2h}{a}\right)U_2 = 0. \tag{2.04}$$

As the independent variables are here accepted instead of r and ϑ values h (height) and s (horizontal distance). Equation for W_2 in our approximation/approach will take the same form, namely

$$\frac{\partial^2 W_2}{\partial h^2} - 2ik\frac{\partial W_2}{\partial s} + k^2\left(\varepsilon - 1 + \frac{2h}{a}\right)W_2 = 0. \tag{2.05}$$

Equations (2.04) and (2.05) we will call the parabolic equations of Leontovich.

During the composition of limiting conditions on the earth's surface (h=0) we can disregard/neglect difference between the dielectric constant of air and unity.

On the other hand, we can somewhat refine these conditions, using our results, obtained according to the method of the addition of the series/rows (see chapters 12 and 6).

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This refinement is reduced to the fact that in (1.07) we replace η and $\eta+1$, and in (1.08) we replace η on $\eta-1$. As a result we will obtain

$$\frac{\partial U_2}{\partial h} = -\frac{ik}{\sqrt{\eta+1}}U_2 \qquad (\text{при}^{(1)}\ h=0) \qquad (2.06)$$

and

$$\frac{\partial W_2}{\partial h} = -ik\sqrt{\eta-1}\,W_2 \qquad (\text{при}\ h=0). \qquad (2.07)$$

Key: (1). with.

Further we must formulate requirement so that in the region near the source where it is possible to disregard the curvature of the earth's surface and the curvature of ray/beam, would occur reflecting formula

for plane earth. If the height/altitude of the source above the earth/ground exists h'=b-a, then this requirement means that in the region indicated must be

$$U_2 = \sqrt{\frac{a}{s}}\left\{e^{i\frac{k(h-h')^2}{2s}} + e^{i\frac{k(h+h')^2}{2s}}\,\frac{h+h'-\frac{s}{\sqrt{\eta-1}}}{h+h'+\frac{s}{\sqrt{\eta-1}}}\right\} \quad (2.08)$$

and

$$W_2 = \sqrt{\frac{a}{s}}\left\{e^{i\frac{k(h-h')^2}{2s}} - e^{i\frac{k(h+h')^2}{2s}}\,\frac{h+h'-s\sqrt{\eta-1}}{h+h'+s\sqrt{\eta-1}}\right\}. \quad (2.09)$$

Factors with second exponential function represent approximate values of Fresnel's coefficients for the vertical and for the horizontal polarization. Latter/last two formulas are the generalization of formula (1.28) of chapter 13.

Let us note that expressions (2.08) and (2.09) themselves approximately satisfy limiting conditions (2.06) and (2.07).

In the case of the field above absolutely conducting surface ($\eta = \infty$) limiting conditions (2.06) and (2.07) take the form

$$\frac{\partial U_2}{\partial h} = 0 \quad (\text{при}^{(1)}\ h = 0) \quad (2.10)$$

and

$$W_2 = 0 \quad (\text{при}^{(1)}\ h = 0), \quad (2.11)$$

Key: (1). with.

and reflecting formulas will be written in the form

$$U_2 = \sqrt{\frac{a}{s}}\left| e^{\frac{ik(h-h')^2}{2s}} + e^{\frac{ik(h+h')^2}{2s}} \right| \quad (2.12)$$

and

$$W_2 = \sqrt{\frac{a}{s}}\left| e^{\frac{ik(h-h')^2}{2s}} - e^{\frac{ik(h+h')^2}{2s}} \right|. \quad (2.13)$$

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3. Analogy with the unsteady task of quantum mechanics.

The problem formulated in the preceding/previous paragraph about the propagation of wave in the spherical layer with the variable/alternating refractive index presents analogy with the quantum-mechanical problem about the motion of wave packet in the assigned force field.

Let us write the equation of Schroedinger for the particle motion of mass m_0 in the force field with potential energy Φ. Designating through x the coordinate of particle, after t - the time, through $\hbar$ - divided on 2π Planck's constant, we will have

$$\frac{\partial^2\psi}{\partial x^2} + 2i\frac{m_0}{\hbar}\frac{\partial\psi}{\partial t} - \frac{2m_0}{\hbar^2}\Phi\psi = 0. \quad (3.01)$$

Comparing the equation of Schroedinger (3.01) with the equation of Leontovich (2.04) or (2.05) for U_2 and for W_2, we see that these equations take one and the same form, moreover coordinate x is proportional to height/altitude h, time t is proportional to the horizontal distance of s and potential energy Φ is proportional to

value $\varepsilon - 1 \div \frac{2h}{a}$, undertaken with the opposite sign which is characterized by from the so-called given (or modified) refractive index

$$M = 10^6\left(\frac{\varepsilon - 1}{2} + \frac{h}{a}\right) = 10^6\left(n - 1 \div \frac{h}{a}\right) \qquad (3.02)$$

only constant factor.

Thus, the parabolic equation of Leontovich for the amplitude of stationary process coincides in form with the unsteady equation of Schroedinger.

The resemblance between both problems is not limited to the coincidence of differential equations, but it is spread also to the maximum and "initial" conditions.

To the case of self-adjoint differential equations and limiting conditions in question in quantum mechanics corresponds in the electromagnetic problem the case of the absence of absorption in the air and in the earth/ground, i.e., that case wher the refractive index of air is real, and the earth/ground - absolute conductor. This case is of greatest interest, also, for the problem about the superrefraction. However, the quantum-mechanical methods can be generalized, also, in the case of the presence of absorption.

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If the earth/ground - absolute conductor, then limiting conditions for U_2 and for W_2 take form (2.10) and (2.11), and to them they correspond in the quantum-mechanical problem of the condition

$$\frac{\partial\psi}{\partial x}=0 \quad (\overset{(1)}{\text{при}}\ x=0) \qquad (3.03)$$

Key: (1). with.

or

$$\psi=0 \quad (\overset{(1)}{\text{при}}\ x=0). \qquad (3.04)$$

Key: (1). with.

However, as far as initial condit-'ons are concerned, their general view consists of the assignment of the initial value of the wave function

$$\psi=\psi_0(x) \quad (\overset{(1)}{\text{при}}\ t=0,\ 0<x<\infty). \qquad (3.05)$$

Key: (1). with.

Function ψ, which satisfies differential equation, initial and limiting conditions, can be searched for in the form

$$\psi(x,t)=\int_0^\infty F(x,x',t)\psi_0(x')\,dx'. \qquad (3.06)$$

Function F must with everyone x' satisfy the differential equation

$$\frac{\partial^2F}{\partial x^2}+2i\frac{m_0}{\hbar}\frac{\partial F}{\partial t}-\frac{2m_0}{\hbar^2}\Phi F=0 \qquad (3.07)$$

and the limiting condition of form (3.03) or (3.04) (to the same as ψ)

So that expression (3.06) with t=0 would be led to $\psi_0(x)$, function F must with t→0 have a special feature/peculiarity whose character was connected with the limiting condition. In the case of the condition

$$\frac{\partial F}{\partial x}=0 \quad (\text{при}^{(1)}\ x=0) \qquad (3.08)$$

Key: (1). with.

special feature/peculiarity F must be the form

$$F(x,x',t)=e^{-i\frac{\pi}{4}}\sqrt{\frac{m_0}{2\pi\hbar t}}\left(e^{i\frac{m_0(x-x')^2}{2\hbar t}}+e^{i\frac{m_0(x+x')^2}{2\hbar t}}\right). \qquad (3.09)$$

However, in the case of the condition

$$F=0 \quad (\text{при}^{(1)}\ x=0) \qquad (3.10)$$

Key: (1). with.

special feature/peculiarity F must be the form

$$F(x,x',t)=e^{-i\frac{\pi}{4}}\sqrt{\frac{m_0}{2\pi\hbar t}}\left(e^{i\frac{m_0(x-x')^2}{2\hbar t}}-e^{i\frac{m_0(x+x')^2}{2\hbar t}}\right). \qquad (3.11)$$

Comparing these formulas with (2.12) and (2.13), we see that for the corresponding limiting conditions special feature/peculiarity F in the accuracy coincides with special features/peculiarities of U_2 and W_2. Actually/really, equalizing height/altitude h to coordinate x, we must assume

$$h=x;\ h'=x',\ \frac{s}{k}=\frac{\hbar t}{m_0}. \qquad (3.12)$$

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Expressing F through the variable/alternating h, h' s, we will have for limiting condition (3.08)

$$F = F_2(h, h', s), \qquad (3.13)$$

where F_2 satisfies the same equation, as U_2, to the limiting condition

$$\frac{\partial F_2}{\partial h} = 0 \quad (\overset{(1)}{\text{при}}\ h = 0) \qquad (3.14)$$

Key: (1). with.

and has the special feature/peculiarity

$$F_2(h, h', s) = e^{-i\frac{\pi}{4}} \sqrt{\frac{k}{2\pi s}} \left(e^{i\frac{k(h-h')^2}{2s}} - e^{i\frac{k(h-h')^2}{2s}}\right). \qquad (3.15)$$

For limiting condition (3.10) we assume/set

$$F = G_2(h, h' s). \qquad (3.16)$$

where G_2 satisfies the same differential equation, as W_2, to the limiting condition

$$G_2 = 0 \quad (\overset{(1)}{\text{при}}\ h = 0) \qquad (3.17)$$

key: (1). with.

and has the special feature/peculiarity

$$G_2(h, h', s) = e^{-i\frac{\pi}{4}} \sqrt{\frac{k}{2\pi s}} \left(e^{i\frac{k(h-h')^2}{2s}} - e^{i\frac{k(h+h')^2}{2s}}\right). \qquad (3.18)$$

We see that F_2 differs only in terms of constant factor from U_2, and G_2 - from W_2, namely we have:

$$U_2 = e^{i\frac{\pi}{4}} \sqrt{\frac{k}{2\pi a}} F_2 \qquad (3.19)$$

and

$$W_2 = e^{i\frac{\pi}{4}} \sqrt{\frac{k}{2\pi a}} G_2. \qquad (3.20)$$

If we designate through f(h, s) the function, which satisfies the same equation and the same limiting conditions as U_2, and taking with s=0 the value

$$f(h,\ s) = f_0(h) \quad (\overset{(1)}{\text{при}}\ s = 0). \qquad (3.21)$$

Key: (1). with.

that on the basis (3.19) we can write

$$f(h,\ s) = e^{-i\frac{\pi}{4}} \sqrt{\frac{k}{2\pi a}} \int_0^\infty U_2(h,\ h',\ s)\, f_0(h')\, dh'. \qquad (3.22)$$

It is analogous, if f(h, s) satisfies the same limiting conditions that also W_2, will be

$$f(h,\ s) = e^{-i\frac{\pi}{4}} \sqrt{\frac{k}{2\pi a}} \int_0^\infty W_2(h,\ h',\ s)\, f_0(h')\, dh'. \qquad (3.23)$$

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Latter/last two formulas are valid not only for the limiting conditions, which correspond absolutely to the conducting earth/ground (when there is an analogy with quantum mechanics), but also for more general/more common/more total maximum conditions (2.06) and (2.07), moreover then special features/peculiarities U_2 and W_2 are given by formulas (2.08) and (2.09).

If function $f_0(h)$ is different from zero only in the region point h=h', moreover integral of f_0 on this region is a value final,

then f(h, s) will be, with not too small s, is proportional U_2 or respectively W_2. Thus, functions U_2 and W_2 correspond to point source, arranged/located on height/altitude h' as this and must be.

In the language of quantum mechanics it is possible to say that function φ, proportional U_2 or W_2, represents the solution of the problem about the deliquescence of wave packet, initially concentrated in the vicinities of one point.

From quantum mechanics it is known that the speed of deliquescence significantly depends on the form of potential energy. Let us visualize that the particle motion is limited on the one hand by the impenetrable wall. If potential energy is such, that the force always is directed from the wall, then deliquescence occurs rapidly. But if force holds particles in certain region where the potential energy has a minimum, or near the wall, against the deliquescence it occurs slowly or does not occur completely. In this case the equation of Schroedinger admits the solutions, which correspond to steady states.

The wave function of almost steady state at zero time is different from zero only in the region of the minimum of potential energy. In the course of time the amplitude of wave function in this region decreases and occurs the decomposition of initial almost

steady state. The decrease of amplitude occurs according to the exponential law, and the speed of decomposition is characterized by coefficient in the index which is called of disintegration constant.

If initial wave function itself is not the wave function of almost steady state, then in the resolution of it it is possible to isolate member, who corresponds to almost steady state, and for the high values of time this member will be the main thing.

In our electromagnetic problem the horizontal distance s plays the role of time t of the quantum-mechanical problem. To the deliquescence of wave packet corresponds the decrease of the amplitude of field with an increase in the horizontal distance. The role of wall plays earth's surface h=0.

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Wall will be impenetrable, if the earth/ground is absolute conductor; with the final conductivity the wall will be absorbing and the decrease of amplitude will occur not only as a result of wave leak into the upper air, but also as a result of absorption by its earth/ground. The role of potential energy plays, as we saw, the reduced refractive index M undertaken with the opposite sign. The course of the given refractive index depending on height/altitude is

shown schematically in Fig. 1.

Unbroken curve showed course M in the presence of superrefraction. The broken continuation of rectilinear segment of a curve corresponds to that case when there is no superrefraction and it is possible to introduce an "equivalent radius" of the Earth, straight line proportional to angular coefficient with respect to axis M.

If we consider the curve of Fig. 1 as the curve of potential energy, then it will be clear that the characteristic for the superrefraction presence of maximum for (-M) (minimum for M) is the necessary condition for existence almost steady state. Actually/really, if we designate through h_m the height/altitude, which corresponds to the maximum potential energy, then in region $h < h_m$ force will seemingly force wave packet against wall and not give to it to depart to region $h > h_m$.

But in our electromagnetic problem the presence of almost steady state indicates such propagation of wave, in which its amplitude decreases with an increase in the distance anomalously slowly, so that attenuation factor of its (corresponding to disintegration constant) is anomalously low. Hence it follows that the condition for existence of almost steady states - this is the condition of the

possibility of hyperdistant radiowave propagation.

The analogy given here with quantum mechanics makes it possible to compose the qualitative picture of the phenomenon of hyperdistant radiowave propagation. This analogy is useful for those that on its basis some, used in quantum mechanics, the mathematical methods can be transferred into the region of radiophysics.

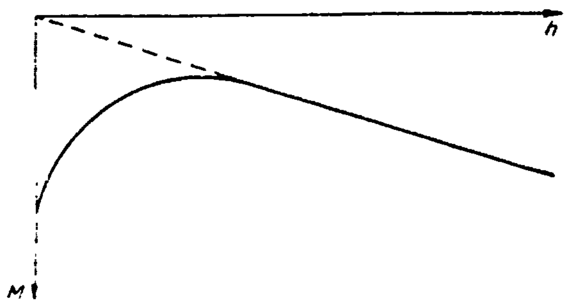

Fig. 1. Given refractive index during the superrefraction.

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On the other hand, developed by us the methods of the solution of the problem about the radiowave propagation can find use, also, in quantum mechanics [1].

FOOTNOTE [1]. Thus, the contemporary theory of the composite moment of momentum is directly connected with our methods of the addition of series/rows, based on the integration in the plane of complex variable ν (chapter 10). (Note, introduced into this edition). ENDFOOTNOTE.

However, these questions emerge from the scope of this book.

4. Transition to the dimensionless quantities.

Let us return to the solution of the problem, formulated in paragraph 2. We should determine functions U_2 and W_2, which satisfy differential equations (2.04) and (2.05), to limiting conditions (2.06) and (2.07) and to conditions (2.08) and (2.09), that characterizes special feature/peculiarity. This problem have solved we earlier for two cases: a) unhomogeneous atmosphere, source on the earth/ground and b) homogeneous atmosphere, source was elevated. We will now show that this problem can be solved and for the general case of unhomogeneous atmosphere and elevated source.

Let us switch over in our equations to the dimensionless quantities, used in the preceding/previous chapters. For this let us examine coefficient with U_2 in equation (2.04). This coefficient is proportional to value

$$\frac{\varepsilon-1}{2}+\frac{h}{a}=10^{-6}M(h). \qquad (4.01)$$

where M(h) - the "modified" refractive index.

We let us assume that, beginning from certain height/altitude h=H, this value it is possible to approximate by linear function from h and to place

$$\frac{\varepsilon-1}{2}+\frac{h}{a}=\alpha+\frac{h}{a^*}, \qquad (4.02)$$

where a* - the so-called equivalent radius of the Earth and α - certain low constant (for example, α<0.0005).

In the simplest case it is possible to consider that with h>H (where H - certain height/altitude(will be $\varepsilon = 1$; then it is necessary to assume α=0 and a*=a.

In that region where is valid formula (4.02), equation for U_2 takes the form

$$\frac{\partial^2 U_2}{\partial h^2} + 2ik \frac{\partial U_2}{\partial s} + k^2 \left(2\alpha + \frac{2h}{a^*}\right) U_2 = 0. \qquad (4.03)$$

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In order to get rid of value α in the latter/last term, let us produce in it the substitution

$$U_2 = Ce^{i\alpha ks} \Psi, \qquad (4.04)$$

where C - constant by which we will be ordered subsequently.

Then equation (4.03) is reduced to the form

$$\frac{\partial^2 \Psi}{\partial h^2} + 2ik \frac{\partial \Psi}{\partial s} + k^2 \frac{2h}{a^*} \Psi = 0. \qquad (4.05)$$

Let us introduce the abbreviation

$$m = \left(\frac{ka^*}{2}\right)^{1/3} \qquad (4.06)$$

and let us assume

$$ks = 2m^2 x, \; kh = my, \; kh' = my'. \qquad (4.07)$$

Then equation (4.05) is written in the form

$$\frac{\partial^2 \Psi}{\partial y^2} + i \frac{\partial \Psi}{\partial x} + y\Psi = 0. \qquad (4.08)$$

The same substitutions reduce more precise equation (2.04) to the

form

$$\frac{\partial^2 \Psi}{\partial y^2} - i\,\frac{\partial \Psi}{\partial x} + [y - r(y)]\,\Psi = 0, \qquad (4.09)$$

where

$$r(y) = m^2\left(\varepsilon - 1 + \frac{2h}{a} - 2\alpha - \frac{2h}{a^*}\right). \qquad (4.10)$$

Value r(y) characterizes the anomalous course of refractive index near the earth's surface; beginning from certain value of y, value r(y) it can be placed equal to zero. If we consider that α=0 and a*=a, then

$$r(y) = m^2(\varepsilon - 1). \qquad (4.11)$$

To us it remains to express under new the variable/alternating limiting conditions and the conditions, which characterize special feature/peculiarity. Assuming/setting

$$q = \frac{im}{\sqrt{\eta + 1}}, \qquad (4.12)$$

we will have

$$\frac{\partial \Psi}{\partial y} - q\Psi = 0 \quad (\overset{(1)}{\text{при}}\ y = 0). \qquad (4.13)$$

Key: (1). with.

The constant C in equation (4.04) we will select so that the equation, analogous (3.22), could be written in the form

$$f(x, y) = \int_0^\infty \Psi(x, y, y')\, f_0(y')\, dy'. \qquad (4.14)$$

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Then the equation, which is determining special feature/peculiarity Ψ, takes the form

$$\Psi = \frac{e^{-i\frac{\pi}{4}}}{2\sqrt{\pi x}}\left|e^{i\frac{(y-y')^2}{4x}} - e^{i\frac{(y+y')^2}{4x}}\frac{y+y'-2iqx}{y-y'-2iqx}\right|. \quad (4.15)$$

From comparison (2.06) with (4.15) we obtain

$$C = e^{i\frac{\pi}{4}}\frac{\sqrt{2\pi ka}}{m}. \quad (4.16)$$

Function W_2 differs from U_2 only in terms of the fact that under the limiting conditions and in the equation, which is determining special feature/peculiarity, value $\frac{1}{\sqrt{\eta+1}}$ is replaced on $\sqrt{\eta-1}$. This corresponds to the replacement of value q on

$$q_1 = im\sqrt{\eta-1}. \quad (4.17)$$

Virtually it is possible in all cases to assume/set $q_1=\infty$.

Together with function Ψ we will examine the function

$$V(x, y, y', q) = 2\sqrt{\pi x}\, e^{i\frac{\pi}{4}}\Psi. \quad (4.18)$$

which we will call attenuation factor. Values U_2 and U are expressed as V as follows:

$$U_2 = e^{i\alpha ks}\sqrt{\frac{a}{s}}\,V \quad (4.19)$$

and

$$U = \frac{e^{i(1+\alpha)ks}}{\sqrt{sa\sin\frac{s}{a}}}V(x, y, y', q). \quad (4.20)$$

Function W is obtained from (4.20) by replacement q on q_1.

5. Properties of the particular solutions of differential equation.

For the construction of function Ψ, of that satisfying the

conditions presented, it is necessary to investigate the properties of the particular solutions of equation (4.09), obtained by separation of variables. Assuming/setting

$$\Psi = e^{ixt} f(y, t), \tag{5.01}$$

we obtain for f(y, t) the equation

$$\frac{d^2 f}{dy^2} + [y + r(y) - t] f = 0. \tag{5.02}$$

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Let us designate through $f^0(y, t)$ and $f^*(y, t)$ the solution of equation (5.02), which satisfy the initial conditions

$$f^0(0, t) = 1, \quad \left(\frac{\partial f^0}{\partial y}\right)_{y=0} = 0 \qquad (5.03)$$

and

$$f^*(0, t) = 0, \quad \left(\frac{\partial f^*}{\partial y}\right)_{y=0} = 1. \qquad (5.04)$$

The general solution of equation (5.02) will take the form

$$f(y, t) = A^0 f^0(y, t) + A^* f^*(y, t). \qquad (5.05)$$

On the other hand, if function r(y) to increases y decreases sufficiently rapidly, then with real t equation (5.02) will have one integral (definition with an accuracy to the factor, which does not depend on y), which behaves with large y as $w_1(t-y)$, and another integral which behaves as $w_2(t-y)$, where w_1 and w_2 - composite Airy's functions, which represent the solutions of the equation

$$\frac{d^2w}{dy^2} + (y-t)w = 0, \qquad (5.06)$$

obtained from (5.02) by replacement r(y) by zero. Functions w_1 and w_2 have the asymptotic expressions

$$w_1(t-y) = e^{i\frac{\pi}{4}}(y-t)^{-\frac{1}{4}}e^{i\frac{2}{3}(y-t)^{3/2}} \quad (5.07)$$

and

$$w_2(t-y) = e^{-i\frac{\pi}{4}}(y-t)^{-\frac{1}{4}}e^{-i\frac{2}{3}(y-t)^{3/2}}. \quad (5.08)$$

Therefore the behavior of the general solution of equation (5.02) with $y \to \infty$ and real t can be characterized by the constants C_1 and C_2 in the expression

$$f(y, t) = C_1 w_1(t-y) + C_2 w_2(t-y). \quad (5.09)$$

Let us establish connection/communication between the constants A°, A^*, C_1 and C_2 (which can be functions from parameter t).

In view of equations (5.02) and (5.06) we have:

$$\frac{d}{dy}\left(w\frac{df}{dy} - f\frac{dw}{dy}\right) = -r(y) f w(t-y). \quad (5.10)$$

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In this equality we can assume consecutively/serially $w=w_1$, then $w=w_2$ and integrate it from 0 to ∞. As a result of the relationship/ratio

$$\frac{\partial w_1}{\partial y} w_2 - \frac{\partial w_2}{\partial y} w_1 = 2i \quad (5.11)$$

we will have

$$\lim_{y\to\infty}\left(w_2\frac{df}{dy} - f\frac{dw_2}{dy}\right) = 2iC_1 \quad (5.12)$$

and

$$\lim_{y\to\infty}\left(w_1\frac{df}{dy} - f\frac{dw_1}{dy}\right) = -2iC_2, \quad (5.13)$$

equality (5.10) after integration gives

$$2iC_1 = A^0 w_2'(t) + A^* w_2(t) - \int_0^\infty r(y) f(y, t) w_2(t-y)\,dy \quad (5.14)$$

and

$$-2iC_2 = A^0 w_1'(t) + A^* w_1(t) - \int_0^\infty r(y) f(y, t) w_1(t-y)\, dy. \quad (5.15)$$

If we here instead of f(y, t) substitute expression (5.05), we will obtain unknown connection/communication between the constants A^0, A^*, C_1, C_2 in the form

$$2iC_1 = A^0 \left\{ w_2'(t) - \int_0^\infty r(y) f^0(y, t)\, w_2(t-y)\, dy \right\} +$$
$$+ A^* \left\{ w_2(t) - \int_0^\infty r(y) f^*(y, t)\, w_2(t-y)\, dy \right\} \quad (5.16)$$

and

$$-2iC_2 = A^0 \left\{ w_1'(t) - \int_0^\infty r(y) f^0(y, t)\, w_1(t-y)\, dy \right\} +$$
$$+ A^* \left\{ w_1(t) - \int_0^\infty r(y) f^*(y, t)\, w_1(t-y)\, dy \right\}. \quad (5.17)$$

Let us pay now attention to the fact that the coefficients with A^0 and of A^* in these equations are whole transcendental functions from parameter t. Actually/really, function f^0, f^*, w_1, w_2, they are goal functions from t, whereas integration is conducted actually in the final limits, since r(y) it is possible to assume equal to zero beginning with certain y. (The same conclusion it will be correct and without this limitation for r(y), if only r(y) sufficiently rapidly decreases at infinity).

Hence it follows that if the constants A^0 and A^* will be whole

transcendental functions from t, then the same character will have the constants C_1 and C_2. This makes it possible for us to apply equations (5.16) and (5.17), derived for the case real t, also in the case of the arbitrary composite values t.

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If we will assume

$$A^0 = A_1^0(t) \equiv w_1(t) - \int_0^\infty r(y) f^*(y, t) w_1(t-y)\, dy \qquad (5.18)$$

and

$$A^* = A_1^*(t) \equiv -w_1^*(t) - \int_0^\infty r(y) f^0(y, t) w_1(t-y)\, dy. \qquad (5.19)$$

the expression

$$f_1(y, t) = A_1^0(t) f^0(y, t) - A_1^*(t) f^*(y, t) \qquad (5.20)$$

will be such solution of equation (5.02), which behaves with $y \to \infty$ as $w_1(t-y)$, and it represents at the same time whole transcendental function from t. It is analogous, if we will assume

$$A^0 = A_2^0(t) \equiv w_2(t) - \int_0^\infty r(y) f^*(y, t) w_2(t-y)\, dy \qquad (5.21)$$

and

$$A^* = A_2^*(t) \equiv -w_2^*(t) + \int_0^\infty r(y) f^0(y, t) w_2(t-y)\, dy. \qquad (5.22)$$

the expression

$$f_2(y, t) = A_2^0(t) f^0(y, t) - A_2^*(t) f^*(y, t) \qquad (5.23)$$

will behave with $y \to \infty$ as $w_2(t-y)$, and it will be integral function

from t.

Integral $f_1(y, t)$ will have the asymptotic expression

$$f_1(y, t) = \frac{c' e^{i\frac{\pi}{4}}}{\sqrt[4]{y - t + r(y)}} \exp\left[i \int_{\tau}^{y} \sqrt{u - t - r(u)}\, du\right], \quad (5.24)$$

while integral $f_2(y, t)$ will have the asymptotic expression

$$f_2(y, t) = \frac{c'' e^{-i\frac{\pi}{4}}}{\sqrt[4]{y - t - r(y)}} \exp\left[-i \int_{\tau}^{y} \sqrt{u - t + r(u)}\, du\right], \quad (5.25)$$

where c', c", τ - constants.

If we place $r(y)=0$, $\tau=t$, $c'=c''=1$, then formulas (5.24) and (5.25) will pass into asymptotic expressions (5.07) and (5.08) for w_1 and w_2.

Integral $f_1(y, t)$ we already used in Chapter 13, where, however, it was accepted without the proof that there is such integral, which allows/assumes asymptotic representation (5.24) and is at the same time whole transcendental function from t.

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Formulas (5.18)-(5.23), derived in the course of proof of this confirmation, can be used also for the actual (numerical) construction of this integral.

Function $f_1(y, t)$ with composite t will grow/rise with increase y, and integral of square $f_1(y, t)$, undertaken on y from 0 to ∞, will be divergent. However, under some assumptions about behavior r(y) in the composite plane, function $f_1(y, t)$ will behave with composite y as $w_1(t-y)$, and will vanish on ray/beam $y = re^{i\alpha}$ (where $\alpha = \frac{\pi}{3}$), so that the integral

$$I = \int_0^{\infty e^{i\alpha}} f_1^2(y, t)\,dy \tag{5.26}$$

will already converge. Let us calculate the value of this integral. Differentiating equation (5.02) for t, we will obtain

$$\frac{d^2}{dy^2}\left(\frac{\partial f}{\partial t}\right) - [y - r(y) - t]\frac{\partial f}{\partial t} = f. \tag{5.27}$$

Hence and from (5.02) we obtain the relationship/ratio

$$\left(f\frac{\partial^2 f}{\partial y\,\partial t} - \frac{\partial f}{\partial t}\frac{\partial f}{\partial y}\right)_0^t = \int_0^t f^2\,dy. \tag{5.28}$$

Assuming/setting here $f=f_1(y, t)$ and counting upper integration limit by equal to $\infty e^{i\alpha}$, we will have

$$\int_0^{\infty e^{i\alpha}} f_1^2(y, t)\,dy = -\left(f_1\frac{\partial^2 f_1}{\partial y\,\partial t} - \frac{\partial f_1}{\partial t}\frac{\partial f_1}{\partial y}\right)_0. \tag{5.29}$$

6. Construction of solution in the form counter integral and in the form of series/row.

In the preceding/previous paragraph we established the existence of two integrals of the ordinary differential equation

$$\frac{d^2 f}{dy^2} - [y + r(y) - t] f = 0, \qquad (6.01)$$

which are whole transcendental functions from parameter t and have asymptotic expressions (5.24) and (5.25). Integrals these, designated by us through $f_1(y, t)$ and $f_2(y, t)$ are determined by formulas (5.20) and (5.23).

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We will show now that with the aid of functions f_1 and f_2 it is possible to construct for the V and for Ψ contour integral, which is the solution of our problem. Reasonings our it will be analogous to the reasonings, presented in the paragraph 3 of Chapter 13, and final formulas will present analogy to formulas (2.24) and (3.10) Chapter 12.

Let us designate through $D_{12}(t)$ Wronskian:

$$D_{12}(t) = f_1 \frac{\partial f_2}{\partial y} - f_2 \frac{\partial f_1}{\partial y} \qquad (6.02)$$

and it is placed

$$F(t, y, y', q) =$$
$$= \frac{1}{D_{12}(t)} f_1(y', t) \left\{ f_2(y, t) - \frac{f_2'(0, t) + q f_2(0, t)}{f_1'(0, t) + q f_1(0, t)} f_1(y, t) \right\}, \qquad (6.03)$$

where primes with f_1 and f_2 designate derivatives on y.

We will consider that y'>y, and let us compose the integral

$$\Psi = \frac{1}{2\pi i}\int e^{ixt}F(t,\ y,\ y',\ q)\,dt, \qquad (6.04)$$

undertaken on the duct/contour, which covers in the positive direction all of the pole of integrand.

From the determination of function F it follows that it is meromorphic function by the variable/alternating t (i.e. it has with final t of no special features/peculiarities, except poles). Function F is completely determined, even if the entering it functions f_1 and f_2 are determined only with an accuracy to the factors, which do not depend on y. Since integrals f_1 and f_2 with all values t are independent (this is evident from their asymptotic expressions), then Wronskian $D_{12}(t)$ roots does not have, and the unique poles F are the roots of the equation

$$f_1'(0,\ t) + qf_1(0,\ t) = 0. \qquad (6.05)$$

If in differential equation (6.01) function r(y) is equal to zero, then it is possible to assume

$$f_1(y,\ t) = w_1(t - y),\ f_2(y,\ t) = w_2(t - y). \qquad (6.06)$$

Then

$$D_{12}(t) = -2i \qquad (6.07)$$

and expression (6.03) for F is led to that examined in Chapter 12 [formula (2.21)].

Let us show that function Ψ, determined by contour integral (6.04), satisfies all conditions presented.

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It is first of all obvious that function Ψ satisfies differential equation (4.09), since it satisfies expression under the integral. Further, function Ψ satisfies limiting condition (4.13), since with all t and y' we have

$$\frac{\partial F}{\partial y} + qF = 0 \qquad (\text{при } \overset{(1)}{} y = 0). \qquad (6.08)$$

Key: (1). with.

Us it remains to show that function Ψ has necessary special feature/peculiarity.

With the aid of asymptotic expressions (5.24) and (5.25) it is possible to show that if x, y are small, and ratio y/x is great, then the main section of integration in (6.04) will lie/rest at the high negative values of t. But if t is great and is negative, then in differential equation (6.01) main role in the coefficient with f will play term - ·t. Therefore with large negative t we will approximately

have

$$f_1(y, t) \sim f_1(0, t)\, e^{iy\sqrt{-t}} \tag{6.09}$$

and

$$f_2(y, t) \sim f_2(0, t)\, e^{-iy\sqrt{-t}}. \tag{6.10}$$

After substituting these expressions into formula (6.03) for F, we will obtain

$$F = \frac{i}{2\sqrt{-t}} \left\{ e^{i(y'-y)\sqrt{-t}} - \frac{q - i\sqrt{-t}}{q + i\sqrt{-t}} e^{i(y'+y)\sqrt{-t}} \right\}. \tag{6.11}$$

The substitution of this value F into integral (6.04) gives for Ψ the formula of Weyl - van der Polya, who after neglect of the low values (in the second term) is led to expression (4.15), which characterizes special feature/peculiarity Ψ.

Thus, the validity of expression (6.04) for Ψ we have established/installed.

From contour integral (6.04) is not difficult to switch over to the series/row, arranged/located on the deductions, relating to the roots of equation (6.05). Let us write this equation in somewhat more detail.

Utilizing expression (5.20) for $f_1(y, t)$ and initial values (5.03) and (5.04) functions f° and f^{*} we obtain

$$f_1(0, t) = A_1^{0}(t), \quad f_1'(0, t) = A_1^{*}(t) \tag{6.12}$$

and equation (6.05) takes the form

$$A_1^{*}(t)+qA_1^{0}(t)=0. \qquad (6.13)$$

Substituting here values (5.18) and (5.19) functions A°_1 and A^{*}_1, we will have

$$w_1'(t)-qw_1(t)-$$
$$-\int_0^{\infty} r(y)\,[f^0(y,t)-qf^{*}(y,t)]\,w_1(t-y)\,dy=0. \qquad (6.14)$$

This equation we will call characteristic.

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For us it is significant that the left side of the characteristic equation is whole transcendental function from t and that it contains only functions $f^{\circ}(y, t)$ and $f^{*}(y,t)$ which can be obtained for all values of t by the numerical integration of differential equation (5.02) with initial conditions (.03) and (5.04). When function r(y), beginning from certain $y=y_1$, is equal to zero, integration in (6.14) can be carried out, and characteristic equation is reduced to the form

$$w_1'(t-y)\,[f^0(y,t)-qf^{*}(y,t)]+$$
$$-w_1(t-y)\frac{\partial}{\partial y}[f^0(y,t)-qf^{*}(y,t)]=0$$
$$(\text{при}^{(1)}\ y=y_1). \qquad (6.15)$$

Key: (1). with.

Characteristic equation for the case of homogeneous atmosphere takes

the form

$$\omega_1'(t) - q\omega_1(t) = 0. \qquad (6.16)$$

This equation is obtained from the preceding/previous formulas, if we in (6.14) place r(y)=0 or if we in (6.15) assume $y=y_1=0$.

The roots of characteristic equation we will designate through

$$t_1(q),\ t_2(q),\ \ldots \qquad (6.17)$$

These roots will be functions from parameter q.

Let us turn to the calculation of the deductions of integral (6.04). From equations (6.02) and (6.05) it follows that with y=0 and $t=t_s$ will be

$$\frac{f_2'(0, t) - q f_2(0, t)}{D_{12}(t)} = \frac{1}{f_1(0, t)}. \qquad (6.18)$$

Further, derivative on t of the denominator in (6.03) can be represented in the form

$$\frac{\partial^2 f_1}{\partial y\, \partial t} - q\frac{\partial f_1}{\partial t} = f_1 \frac{\partial}{\partial t}\left(\frac{1}{f_1}\frac{\partial f_1}{\partial y}\right) = -f_1(0, t)\frac{dq}{dt}. \qquad (6.19)$$

Therefore the deduction of function F at point $t=t_s$ will be equal to

$$\frac{dt_s}{dq}\frac{f_1(y', t_s)}{f_1(0, t_s)}\frac{f_1(y, t_s)}{f_1(0, t_s)}. \qquad (6.20)$$

Taking the sum of expressions (6.20), multiplied by e^{ixt_s}, we will obtain the unknown expansion of function Ψ in the series/row

$$\Psi = \sum_{s=1}^{\infty} e^{ixt_s} \frac{dt_s}{dq} \frac{f_1(y', t_s)}{f_1(0, t_s)} \frac{f_1(y, t_s)}{f_1(0, t_s)}. \qquad (6.21)$$

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Values

$$\frac{f_1(y, t_s)}{f_1(0, t_s)} = f^0(y, t_s) - qf^*(y, t_s) \qquad (6.22)$$

can be named/called high-altitude factors. Let us note that the high-altitude factors are expressed according to (6.22) through functions f° and f* which are directly calculated by the numerical integration of differential equation (5.02).

When value q is very great or equal to infinity (horizontal polarization, absolute conductor), formula (6.21) must be converted via term-by-term multiplication by value

$$\frac{q^2 f_1^2(0, t_s)}{f_1'^2(0, t_s)} = 1. \qquad (6.23)$$

Result can be written in the form

$$\Psi = \sum_{s=1}^{\infty} e^{ixt_s} \left(q^2 \frac{dt_s}{dq}\right) \frac{f_1(y', t_s)}{f_1'(0, t_s)} \frac{f_1(y, t_s)}{f_1'(0, t_s)}. \qquad (6.24)$$

Value

$$q^2 \frac{dt_s}{dq} = \frac{1}{\frac{d}{dt}\left(\frac{f_1(0, t)}{f_1'(0, t)}\right)} \qquad (6.25)$$

will have with q→∞ finite value. Let us note that from formulas

(6.19) and (5.29) escape/ensues the relationship/ratio

$$f_1^2(0, t)\frac{dq}{dt} = \int_0^{\infty e^{i\alpha}} f_1^2(y, t)\,dy \quad \left(\alpha = \frac{\pi}{3}\right). \qquad (6.26)$$

Therefore series/row (6.21) can be written in the form

$$\Psi = \sum_{s=1}^{\infty} e^{ixt_s} \frac{f_1(y', t_s) f_1(y, t_s)}{\int_0^{\infty e^{i\alpha}} f_1^2(y, t_s)\,dy}. \qquad (6.27)$$

In this form it resembles eigenfunction expansion. In series/row (6.27) the "eigenvalues" are, however, composite and the standardizing integral standing in denominator converges only with the composite way of integration.

In order to switch over from function Ψ to attenuation factor V, it suffices to recollect relationship/ratio (4.18)

$$V(x, y, y', q) = 2\sqrt{\pi x}\, e^{i\frac{\pi}{4}} \Psi.$$

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In the case of homogeneous atmosphere when it is possible to place r(y)=0 and $f_1(y, t)=w_1(t-y)$, expressions escape/ensuing from our formulas for V are led to those such as were derived in Chapter 12 by another method.

7. Use/application of general theory to the case of superrefraction.

(Schematic example).

Examined in the paragraph 3 analogies with the unsteady task of quantum mechanics make it possible to compose for itself the qualitative picture of the phenomenon of superrefraction and those conditions under which this phenomenon can occur. On the other hand, the obtained in paragraph 6 general/common/total expression for attenuation factor is useful for the quantitative calculations which, true, require sufficiently complicated calculations.

Let us write expression for function Ψ connected with the attenuation factor. Assuming/setting for the brevity

$$f(y,t)=f^0(y,t)-qf^*(y,t), \tag{7.01}$$

we will have on the basis (6.21)

$$\Psi=\sum_{s=1}^{\infty} e^{ixt_s}\frac{dt_s}{dq}f(y',t_s)f(y,t_s), \tag{7.02}$$

where of value t_s they are the roots of transcendental equation (6.14).

If r(y)=0 with $y>y_1$, this equation can be according to (6.15) written in the form

$$w_1'(t-y_1)f(y_1,t)+w_1(t-y_1)f'(y_1,t)=0, \tag{7.03}$$

where w'_1 indicates derivative on argument (t-y), and derivative on y. Parameter q enters into this equation through value (7.01).

The determination of the conditions under which the hyperdistant possible propagation, is reduced to the investigation of the roots of characteristic equation(6.14) or (7.03). In the absence of superrefraction the alleged part of the roots of this equation, which according to (7.02) gives weakening wave with an increase in the distance, there will be of the same order as real part. However, in the presence of superrefraction there are one or several roots with the anomalously small alleged part.

In order to compose to itself representation about that, under what conditions can occur hyperdistant propagation, let us examine the following schematic example.

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Let function r(y) take the following form:

$$\begin{aligned} r(y) &= (1+\mu^3)(y_1-y) && (\text{при } 0<y<y_1), \\ r(y) &= 0 && (\text{при } y_1<y). \end{aligned} \qquad (7.04)$$

Key: (1). with.

This corresponds to the assumption that the graph/curve of the given refractive index represents the broken line, depicted in Fig. 2.

If we consider that dielectric constant ε is changed according to the law

$$\left.\begin{aligned} \varepsilon &= 1-g(h-h_1) \quad (\text{при}\ h<h_1), \\ \varepsilon &= 1 \quad (\text{при}\ h>h_1), \end{aligned}\right\} \tag{7.05}$$

Key: (1). with.

that the parameters μ^3 and y_1 will be equal to

$$\mu^3 = \frac{ag}{2} - 1, \quad y_1 = h_1 \sqrt[3]{\frac{2k^2}{a}}. \tag{7.06}$$

Thus, the parameter μ on wavelength does not depend, but parameter y_1 (the reduce height of crank point) will be proportional $\lambda^{-2/3}$.

Equation for f is written in the form

$$\left.\begin{aligned} \frac{d^2f}{dy^2} + [(1+\mu^3)y_1 - \mu^3 y - t]f = 0 \quad (\text{при}\ y<y_1), \\ \frac{d^2f}{dy^2} + (y-t)f = 0 \quad (\text{при}\ y>y_1). \end{aligned}\right| \tag{7.07}$$

Key: (1). with.

Let us introduce instead of t the new parameter

$$\xi_0 = \frac{t-(1+\mu^3)y_1}{\mu^2} \tag{7.08}$$

and instead of y - new variable/alternating

$$\xi = \xi_0 + \mu y. \tag{7.09}$$

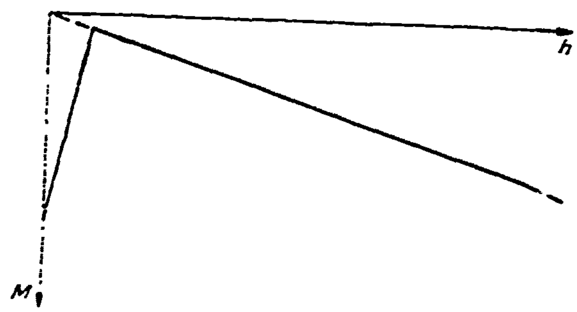

Fig. 2. Schematic course of the given refractive index.

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To value of $y=y_1$ will correspond value $\xi=\xi_1$, where

$$\mu^2\xi_1 = t - y_1. \tag{7.10}$$

Equation (7.05) will take the form

$$\frac{d^2 f}{d\xi^2} = \xi f \quad (\xi_0 < \xi < \xi_1). \tag{7.11}$$

Its independent solutions it will be Airy's function $u(\xi)$ and $v(\xi)$. Functions f° and f^* will beequal to

and

$$\left.\begin{aligned} f^{\circ}(y) &= u'(\xi_0)\, v(\xi) - v'(\xi_0)\, u(\xi) \\ f^*(y) &= \frac{1}{\mu}\left[v(\xi_0)\, u(\xi) - u(\xi_0)\, v(\xi)\right]. \end{aligned}\right| \tag{7.12}$$

In view of the relationship/ratio

$$u'(\xi)\, v(\xi) - v'(\xi)\, u(\xi) = 1 \tag{7.13}$$

of function f° and f^* they will satisfy initial conditions (5.03) and (5.04). Introducing according to (7.01) the function

$$f(y) = -\frac{1}{\mu}[qv(\xi_0) - \mu v'(\xi_0)]\,u(\xi) +$$

$$+\frac{1}{\mu}[qu(\xi_0) - \mu u'(\xi_0)]\,v(\xi), \qquad (7.14)$$

we will obtain characteristic equation, if we substitute values of f(y) and f'(y) with $y=y_1$ into formula (7.03). This characteristic equation can be written in the form

$$\frac{\mu v'(\xi_0) - qv(\xi_0)}{\mu u'(\xi_0) - qu(\xi_0)} = \frac{\mu v'(\xi_1)\,w_1(\mu^2\xi_1) - v(\xi_1)\,w_1'(\mu^2\xi_1)}{\mu u'(\xi_1)\,w_1(\mu^2\xi_1) + u(\xi_1)\,w_1'(\mu^2\xi_1)}. \qquad (7.15)$$

Let us assume that value y_1 and parameter μ is sufficiently great. This means that the "potential pit" in Fig. 2 is sufficiently wide and deep. In that case of value ξ_1 and $\mu^2\xi_1$ [arguments of the functions, entering the right side (7.15)] they will be great. In view of the asymptotic expressions

$$u(\xi) = \xi^{-\frac{1}{4}} e^{\frac{2}{3}\xi^{3/2}}, \quad v(\xi) = \frac{1}{2}\xi^{-\frac{1}{4}} e^{-\frac{2}{3}\xi^{3/2}} \qquad (7.16)$$

the right side of equation (7.15) will be very small, and characteristic equation approximately is led to the following:

$$\mu v'(\xi_0) + qv(\xi_0) = 0. \qquad (7.17)$$

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This the case will occur when in the sufficiently large region of heights/altitudes the gradient of the dielectric constant of air will be negative and greater than 2/a, where a - radius of the Earth; then the curvature of ray/beam will be more than the curvature of the

Earth and formally computed "equivalent radius" is negative.

The attenuation of wave with an increase in the horizontal distance is connected with the alleged part of value t, and also, therefore, with the alleged part of value ξ_0; if ξ_0 they was real, fading would be absent. Fading can originate from two reasons: from the absorption in the earth/ground and from wave leak into the uppe air. Absorption in the earth/ground is characterized by the composite parameter q. Equation (7.17) corresponds to that case when fading occurs only due to the absorption in the earth/ground. If we consider the earth/ground as absolute conductor, it is necessary to assume q=0 for the horizontal polarization and q=∞ for the vertical polarization. With q=0 equation (7.17) is reduced to the form

$$v'(\xi_0) = 0 \quad (\text{при}^{(1)}\ q = 0). \qquad (7.18)$$

Key: (1). with.

Its roots will be the real negative numbers

$$\xi_0 = -1{,}019;\ -3{,}248,\ -4{,}820;\ \ldots \qquad (7.19)$$

With q=∞ equation(7.17) takes the form

$$v(\xi_0) = 0 \quad (\text{при}^{(1)}\ q = \infty) \qquad (7.20)$$

Key: (1). with.

and it has by the roots of the number

$$\xi_0 = -2{,}338;\ -4{,}088;\ -5{,}521;\ \ldots \qquad (7.21)$$

Since in these cases of value ξ_0 they are real, the fading is absent.

Equation (7.17) will present a good approximation/approach to (7.15), if value ξ_1 (or its real part) is positive and sufficiently great. Since

$$\xi_1 = \xi_{00} + \mu y_1,$$

the this condition will cease to be made beginning from certain root ξ_0. Therefore a number of roots with a small alleged part will be final.

It can derive the approximation formula for the correction to value ξ_0, obtained via the account of right side (7.15). Let us designate by ξ_{00} the root of equation (7.17), which we will consider as the defective value ξ_0, and through $\Delta\xi_0$ - correction. This correction will be obtained, if we substitute into right side (7.15) approximate value ξ_1, equal to

$$\xi_1 = \xi_{00} + \mu y_1.$$

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Approximate value of correction will be obtained then from the equation

$$\Delta\xi_0 = \frac{1}{1-\xi_{00}}\left\{\frac{1}{16}\frac{\mu^3-1}{\mu^3\xi_1^{3/2}}e^{-\frac{4}{3}\xi_1^{3/2}} \quad \frac{i}{4}e^{-2S}\right\}. \qquad (7.22)$$

where

$$S = \frac{2}{3}(\mu^3+1)\xi_1^{3/2}. \qquad (7.23)$$

Let us note that the alleged part of the correction is positive. This corresponds to that fact that the leakage into the upper layers increases fading.

Applicability condition for these approximation formulas is sufficiently high value μy_1. Let us recall that according to (7.06) we have

$$\mu y_1 = h_1 \sqrt[3]{k^2\left(g-\frac{2}{a}\right)}, \qquad (7.24)$$

where g - undertaken with the opposite sign gradient of dielectric constant; a - radius of the Earth and h_1 - spot height of fracture in Fig. 2.

The greater the value μy_1, the greater the number of almost steady states with a small fading. Tentatively it is possible to say that a number of such states is equal to a number of roots ξ_0, which do not exceed (in the absolute value) the parameter μy_1.

The concept about the ray/beam, which is reflected from the

upper bound of layer and from the earth's surface, begins to become applicable if and only if a number of almost steady states [number of terms of series/row (7.02) with a small fading] it becomes large. Generally, the necessary condition for the applicabiliteof the concepts of geometric optic/optics is slow convergence of series (7.02), when in it plays a role a large number of members. But if in it are essential one-two members (which can correspond both almost to stationary and damped states), then the concept of ray/beam is applicable completely.

8. Approximation formulas for the terms with a small fading.

Using method, by the analogous to the volume, such as it is applied in quantum mechanics, it is possible to derive approximation for the high-altitude factors, which correspond to terms with a small fading, and to also give the estimation of that part of the attenuation factor, which corresponds to leakage into the upper layers.

Let us assume in equation (6.01)

$$y + r(y) = p(y) \tag{8.01}$$

and let us write this equation in the form

$$\frac{d^2 f}{dy^2} + [p(y) - t] f = 0. \tag{8.02}$$

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In the case of superrefraction function p(y), proportional to the given refractive index, it will have a minimum, and on both sides from it it will grow/rise; to the left of the minimum the greatest value p(y) will be p(0), and to the right p(y) will increase as does value y. If parameter t lies/rests between the least by value p(y) and by value p(0), then coefficient with f into equation (8.02) will become zero at two values of y which we will designate through y_1 and y_2. In gap/interval $y_1<y<y_2$ value p(y)-t will be negative, and out of this gap/interval - positive.

In gap/interval $y_1<y<y_2$ the solution of equation (8.02) can be approximately expressed through the Airy's functions. Let us assume

$$\int_{y_1}^{y} \sqrt{t-p(y)}\,dy = \frac{2}{3}\xi_1^{3/2} \qquad (8.03)$$

and

$$\int_{y}^{y_2} \sqrt{t-p(y)}\,dy = \frac{2}{3}\xi_2^{3/2} \qquad (8.04)$$

and designate through S sum of these values which does not depend on y:

$$S = \int_{y_1}^{y_2} \sqrt{t-p(y)}\,dy. \qquad (8.05)$$

Value S we will consider it large. With such designations we will

approximately have

$$f = \sqrt[4]{\frac{\xi_1}{t - p(y)}}\,[A_1 u(\xi_1) + B_1 v(\xi_1)]. \tag{8.06}$$

and also

$$f = \sqrt[4]{\frac{\xi_2}{t - p(y)}}\,[A_2 u(\xi_2) + B_2 v(\xi_2)]. \tag{8.07}$$

moreover

$$\frac{2}{3}\xi_1^{3/2} + \frac{2}{3}\xi_2^{3/2} = S \tag{8.08}$$

and the constants A_1, B_1, A_2, B_2 are connected with the relationships/ratios

$$A_2 = \frac{1}{2} B_1 e^{-S}, \quad B_2 = 2A_1 e^{S}, \tag{8.09}$$

which escape/ensue from the comparison of asymptotic expressions for (8.06) and (8.07) at the high values ξ_1 and ξ_2.

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With $y>y_2$ we can determine valu ξ_2 by means of the equality

$$\int_{y_2}^{y} \sqrt{p(y) - t}\, dy = \frac{2}{3}(-\xi_2)^{3/2} \tag{8.10}$$

and use previous expression (8.07) for f.

It is analogous, with $y<y_1$ we can instead of (8.03) assume

$$\int_{y}^{y_1} \sqrt{p(y) - t}\, dy = \frac{2}{3}(-\xi_1)^{3/2} \tag{8.11}$$

and apply for f expression (8.06).

Let us select the constants A, B so that the function f would be proportional $f_1(y, t)$. We must assume

$$A_2 = C_1, \quad B_2 = iC_1 \tag{8.12}$$

and, therefore,

$$A_1 = \frac{i}{2} C_1 e^{-S}, \quad B_1 = 2C_1 e^{S}. \tag{8.13}$$

Then formulas (8.06) and (8.07) take the form

$$f_1(y, t) = 2C_1 e^{S} \sqrt[4]{\frac{\xi_1}{t - p(y)}} \left[v(\xi_1) - \frac{i}{4} e^{-2S} u(\xi_1) \right] \tag{8.14}$$

and

$$f_1(y, t) = C_1 \sqrt[4]{\frac{\xi_1}{t - p(y)}} w_1(\xi_1). \tag{8.15}$$

Analogously are obtained the following approximations for $f_2(y, t)$:

$$f_2(y, t) = 2C_2 e^{S} \sqrt[4]{\frac{\xi_1}{t - p(y)}} \left[v(\xi_1) - \frac{i}{4} e^{-2S} u(\xi_1) \right] \tag{8.16}$$

and

$$f_2(y, t) = C_2 \sqrt[4]{\frac{\xi_2}{t - p(y)}} w_2(\xi_2). \tag{8.17}$$

In this approximation/approach the Wronskian $D_{12}(t)$ proves to be equal to

$$D_{12}(t) = -2iC_1 C_2. \tag{8.18}$$

With $y<y_1$ functions $v(\xi_1)$ and $u(\xi_1)$ will be one order.

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As a result of the smallness of factor e^{-2S} the second terms in (8.14) and (8.16) will present small corrections [generally speaking it is less than an error in entire expression (8.14) or (8.16)]. Therefore in region $y<y_1$ functions f_1 and f_2 will be almost proportional to each other.

Throwing/rejecting small corrections, it is possible to write equation for determining of t in the form

$$\left(\frac{d\xi}{dy}\right)_0 v'(\xi_0) - qv(\xi_0) = 0. \qquad (8.19)$$

We designated here through ξ_0 and $\left(\frac{d\xi}{dy}\right)_0$ the values ξ_1 and $\frac{d\xi}{dy}$ with y=0.

This equation is analogous with equation (7.17). It gives only that part of the attenuation factor of wave, which originates from absorption in the earth/ground. Since the composite parameter q, which characterizes the properties of soil, is known only very roughly, then coefficient $\left(\frac{d\xi}{dy}\right)_0$ it suffices to take in the rough approximation and to assume accordingly (7.06) and (7.09)

$$\left(\frac{d\xi}{dy}\right)_0 = \mu = \sqrt[3]{\frac{ag}{2} - 1}, \qquad (8.20)$$

where a - radius of the Earth and g - undertaken with the opposite sign gradient of dielectric constant.

Then equation (8.19) is reduced to form (7.17), investigated in the preceding/previous paragraph. The roots ξ_0 of equation (8.19)

will be connected with the appropriate values of parameter t with the relationship/ratio

$$k \int_{0}^{h^*_1} \sqrt{-\frac{t}{m^2} - \varepsilon - 1 - \frac{2h}{a}}\, dh = \frac{2}{3} (-\xi_0)^{3/2}, \qquad (8.21)$$

where h^*_1 - smaller of those two values of h^*_1 and h^*_2 height/altitude h, at which a subradical quantity becomes zero.

If ξ_0 it is real, then h^*_1 and t are real; if ξ_0 it is composite, then the calculation of integral (8.21) requires the analytical continuation of the interpolation of formulas for ε into complex domain.

The necessary condition for the applicability of the preceding/previous formulas is the smallness of value e^{-2S}, where S has value (8.05). In usual unity the integral, which expresses S, will be written in the form

$$S = k \int_{h^*_1}^{h^*_2} \sqrt{\frac{t}{m^2} - \left(\varepsilon - 1 + \frac{2h}{a}\right)}\, dh. \qquad (8.22)$$

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After determining t from (8.21), it is necessary to check that integral S for this t is sufficiently great.

In the case of absolute conductor ($q=0$ and $q=\infty$) approximate values ξ_0 and t are from (8.19) and (8.21) real. In this case it is possible to indicate approximate value of the alleged part of the correction to ξ_0.

Assuming/setting

$$\xi_0 = \xi_0' - i\xi_0'', \tag{8.23}$$

we will have

$$\sqrt{1-\xi_0'}\,\xi_0'' = \frac{1}{4}e^{-2S}. \tag{8.24}$$

On the conclusion/output this formula we stop will not be.

Since ξ_0'' - low value, then to increase $\Delta\xi_0 = i\xi_0''$ will correspond increase $\Delta t = it'' = \frac{dt}{d\xi_0}\Delta\xi_0$. But value (8.24) multiplied by i, presents an increase in integral (8.21). Therefore we can determine t" (alleged part t) from the equation

$$t''\frac{\partial}{\partial t}\left(k\int_0^{h_0}\sqrt{-\frac{t}{h^2}-\varepsilon-1+\frac{2h}{a}}\,dh\right) = -\frac{1}{4}e^{-2S}. \tag{8.25}$$

Since derivative of integral is negative, then for t" is obtained positive value, which corresponds to fading.

The obtained formulas make it possible to derive also

approximation for value $\frac{dq}{dt}$. According to (6.19) we have

$$\frac{dq}{dt} = -\frac{\partial \ln \Phi_1}{\partial q\, \partial t}. \qquad (8.26)$$

Substituting here value Φ_1 from (8.14) and disregarding low values, we will obtain

$$\frac{dq}{dt} = \left(\frac{d\xi}{dy}\right)_0 \frac{\partial \xi_0}{\partial t} \left(\frac{v'^2(\xi_0)}{v^2(\xi_0)} - \frac{v''(\xi_0)}{v(\xi_0)}\right). \qquad (8.27)$$

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Here we can assume in auto the rough approximation

$$\left(\frac{d\xi}{dy}\right)_0 = \mu, \quad \frac{\partial \xi_0}{\partial t} = \frac{1}{\mu^2} \qquad (8.28)$$

[see formulas (7.08) and (7.09)]. Using differential equation and limiting conditions for v, we will obtain hence

$$\frac{dq}{dt} = \frac{q^2}{\mu^3} - \frac{\xi_0}{\mu}. \qquad (8.29)$$

The first terms of series/row (6.21) for Ψ, possessing a small fading, will be, in our approximation/approach, they are equal to

$$\sum e^{ixt} \frac{v(\xi_1) v(\xi_1')}{\left(\frac{q^2}{\mu^3} - \frac{\xi_0}{\mu}\right) v^2(\xi_0)} = \sum e^{ixt} \frac{\mu v(\xi_1) v(\xi_1')}{v'^2(\xi_0) - \xi_0 v^2(\xi_0)}, \qquad (8.30)$$

where ξ'_1 it relates to reduced height y'.

If ξ_0 it is so great in the absolute value that it is possible to use for $v(\xi_0)$ and $v'(\xi_0)$ asymptotic expressions, then denominator in this formula will be approximately equal to

$$v'^2(\xi_0) - \xi_0 v^2(\xi_0) = 1 - \xi_0. \quad (8.31)$$

In conclusion it is necessary to emphasize that formulas derived in this paragraph are based on the sufficiently rough approximations and are intended for the tentative calculations. More precise calculations must be based on a strict theory, presented in the preceding/previous paragraphs.

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Chapter 15.

Approximation formula for the horizon distance in the presence of superrefraction [1].

FOOTNOTE [1]. Foch, 1956. ENDFOOTNOTE.

On the basis of the general formulas, obtained in Chapter 14, are investigated the cases of normal refraction and superrefraction. For the functions, which stand under the contour integral in the expression for the attenuation factor, are derived the approximations by the asymptotic integration of the corresponding differential equations. In the case of normal refraction these functions are approximated through the Airy's functions, while in the case of superrefraction - through the functions of parabolic cylinder. The qualitative investigation of contour integral leads to the approximation formulas for that distance, beginning from which field rapidly decreases, i.e., for the horizon distance. During the superrefraction when there are many waves reflected, under the "horizon distance" is understood the distance for the first wave reflected. General formulas are applied to the case when the given

refractive index depends on height/altitude according to the hyperboli law.

1. Introduction.

In Chapter 14 is derived for the attenuation factor general formula in the form of contour integral. The obtained expression is applicable for the very general case of the arbitrary course of refractive index depending on height/altitude. Basic difficulty of applying our general formula consists of the solution of differential equation for the high-altitude factor. This of difficulty can be avoided by using the asymptotic solution of the equation (this reception/procedure is based on the presence in equation of the high parameter). After obtaining approximations for the high-altitude factor, it is possible to write in the explicit form integrand in the contour integral and then it to investigate.

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The qualitative investigation of integrand makes it possible to give the estimation of those distances beginning from which attenuation factor begins rapidly to decrease, in other words, the estimation of horizon distance.

2. Initial formulas.

As shown in Chapter 14, field from the vertical and horizontal electrical and magnetic dipole is expressed in the general case through two functions of Hertz: U and W, which satisfy identical differential equations; limiting conditions for U and W also of the identical type, but with the different values of coefficients. Each of the functions of Hertz can be expressed through the attenuation factor V according to the formula

$$U = \frac{e^{iks}}{\sqrt{sa \sin \frac{s}{a}}} V, \qquad (2.01)$$

where a - radius of terrestial globe; s - horizontal distance, counted on the arc of terrestial globe, $k = \frac{2\pi}{\lambda}$ - the absolute value of wave vector.

Attenuation factor the V is most convenient to express as the dimensionless quantities: the given horizontal distance

$$x = \frac{k}{2m^2} s \qquad (2.02)$$

and the reduced height of the corresponding points (source and observation point):

$$y = \frac{k}{m} h, \quad y' = \frac{k}{m} h', \qquad (2.03)$$

where h and h' - heights/altitudes in the units of length, and m - parameter

$$m = \sqrt[3]{\frac{ka}{2}}. \qquad (2.04)$$

In the tasks, connected with the superrefraction, equivalent

radius of the Earth does not play that role such as it plays in the case of normal refraction; therefore we it do not here introduce. Besides the enumerated values, the factor of attenuation V depends on parameter q, entering the limiting conditions. For functioning the Hertz U (vertical polarization) parameter q is equal to

$$q = \frac{im}{\sqrt{\eta - 1}}, \qquad (2.05)$$

where η - composite dielectric constant of medium.

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For functioning the Hertz W (horizontal polarization) parameter q is equal to

$$q = im\sqrt{\eta - 1}. \qquad (2.06)$$

In the latter case it is possible to virtually assume/set q=∞, since parameters m and η are great.

Thus, attenuation factor V is a function from dimensionless quantities x, y, y', q:

$$V = V(x, y, y', q). \qquad (2.07)$$

Together with the attenuation factor V convenient to examine function Ψ, connected with it through which V is expressed according to the formula

$$V = 2\sqrt{\pi x}\, e^{i\frac{\pi}{4}}\Psi. \qquad (2.08)$$

Function Ψ satisfies the differential equation

$$\frac{\partial^2\Psi}{\partial y^2} - i\frac{\partial\Psi}{\partial x} - \left[y + r(y)\right]\Psi = 0, \qquad (2.09)$$

where

$$r(y) = m^2(\varepsilon - 1), \qquad (2.10)$$

moreover $\varepsilon = \varepsilon(h)$ is dielectric constant of air as function from the height/altitude. Equation (2.09) is obtained by transition to the dimensionless quantities from the equation

$$\frac{\partial^2\Psi}{\partial h^2} - 2ik\frac{\partial\Psi}{\partial s} + k^2\left(\frac{2h}{a} + \varepsilon - 1\right)\Psi = 0, \qquad (2.11)$$

in which coefficient with Ψ proportional to the given refractive index

$$M(h) = 10^6\left(\frac{\varepsilon - 1}{2} + \frac{h}{a}\right). \qquad (2.12)$$

Coefficient when Ψ in equation (2.09) it is convenient to designate by one letter and to assume/set

$$p(y) = y + r(y). \qquad (2.13)$$

We have

$$p(y) = m^2\left(\varepsilon - 1 + \frac{2h}{a}\right), \qquad (2.14)$$

so that p(y) there is actually the same given refractive index, that only expressed through the dimensionless height/altitude y.

During the use of designation (2.13) equation (2.09) will be written

$$\frac{\partial^2\Psi}{\partial y^2} + i\frac{\partial\Psi}{\partial x} + p(y)\Psi = 0. \qquad (2.15)$$

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Besides differential equation (2.15), function Ψ satisfies the limiting condition

$$\frac{\partial\Psi}{\partial y} + q\Psi = 0 \quad (\text{при}^{(1)}\ y = 0). \qquad (2.16)$$

Key: (1). with.

With x=0 it has a special feature/peculiarity of form.

$$\Psi = \frac{e^{-i\frac{\pi}{4}}}{2\sqrt{\pi x}}\left\{e^{i\frac{(y-y')^2}{4x}} + e^{i\frac{(y+y')^2}{4x}}\frac{y+y'-2iqx}{y-y'-2iqx}\right\}. \qquad (2.17)$$

In Chapter 14 there was given general/common/total expression for function Ψ in the form of contour integral. Integrand in it is expressed through the solutions of the equation

$$\frac{d^2f}{dy^2} + p(y)f = tf, \qquad (2.18)$$

where t - composite parameter. (These solutions and they were named above high-altitude factors).

For the compositio of integrand it is necessary to know both solutions of equation (2.18); let us designate them through $f_1(y, t)$ and $f_2(y, t)$.

With large y these functions have the asymptotic expressions

$$f_1(y,\ t)=\frac{c' e^{i\frac{\pi}{4}}}{\sqrt[4]{p(y)-t}}\exp\left[i\int_{\tau}^{y}\sqrt{p(u)-t}\,du\right]. \qquad (2.19)$$

$$f_2(y,\ t)=\frac{c'' e^{-i\frac{\pi}{4}}}{\sqrt[4]{p(y)-t}}\exp\left[-i\int_{\tau}^{y}\sqrt{p(u)-t}\,du\right]. \qquad (2.20)$$

Here c', c", τ - constants whose values are unessential, since they drop out from the expression for Ψ. In the case of homogeneous atmosphere when $p(y)=y$, functions $f_1(y,\ t)$ and $f_2(y,\ t)$ are led to the composite Airy's functions $w_1(t-y)$ and $w_2(t-y)$, moreover then it is possible to assume $c'=c''=1$ and $\tau=t$.

Let us assume

$$D_{12}(t)=f_1\frac{\partial f_2}{\partial y}-f_2\frac{\partial f_1}{\partial y}. \qquad (2.21)$$

On the strength of equation (2.18)to which satisfy f_1 and f_2, this value will not depend on y.

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Let us designate values $\frac{\partial f_1}{\partial y}$ and $\frac{\partial f_2}{\partial y}$ with y=0 through $f'_1(0,\ t)$ and $f'_2(0,\ t)$ and will compose the function

$$F(t,\ y,\ y',\ q)=$$

$$=\frac{1}{D_{12}(t)}f_1(y',\ t)\left\{f_2(y,\ t)-\frac{f'_2(0,\ t)+qf_2(0,\ t)}{f'_1(0,\ t)+qf_1(0,\ t)}f_1(y,\ t)\right\}. \qquad (2.22)$$

Function Ψ is determined with $y'>y$ by the contour integral

$$\Psi=\frac{1}{2\pi i}\int e^{ixt}F(t,\ y,\ y',\ q)\,dt. \qquad (2.23)$$

undertaken on the duct/contour, which covers in the positive direction all of the pole of integrand. As shown in Chapter 14, function Ψ satisfies all conditions set above and gives the solution of our problem.

3. Case of normal refraction.

The case of normal refraction is characterized by the fact that the given refractive index M(h) is the monotonically increasing function from height/altitude h, and therefore coefficient p(y) is that monotonically increasing function from y. In this case it is possible to approximately express $f_1(y, t)$ and $f_2(y, t)$ through the composite Airy's functions from the argument ξ, determined by the equalities

$$\int_b^y \sqrt{p(u)-t}\,du = \frac{2}{3}(-\xi)^{3/2}; \quad (3.01)$$

$$\int_y^b \sqrt{t-p(u)}\,du = \frac{2}{3}\xi^{3/2}, \quad (3.02)$$

where b is a root of the equation

$$p(b) = t. \quad (3.03)$$

Near y=b resolution of value ξ according to degrees of y - b it will begin from the linear terms, namely

$$\xi = \sqrt[3]{p'(b)}\,(b-y) + \cdots \tag{3.04}$$

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We can approximately assume

$$f_1 = \sqrt{-\frac{dy}{d\xi}}\,w_1(\xi),\quad f_2 = \sqrt{-\frac{dy}{d\xi}}\,w_2(\xi) \tag{3.05}$$

also, with the same approximation/approach

$$\frac{\partial f_1}{\partial y} = -\sqrt{-\frac{d\xi}{dy}}\,w_1'(\xi),\quad \frac{\partial f_2}{\partial y} = -\sqrt{-\frac{d\xi}{dy}}\,w_2'(\xi), \tag{3.06}$$

whence

$$D_{12} = -2i. \tag{3.07}$$

Replacing here y on y' and ξ on ξ', we will obtain expressions for $f_1(y', t)$ and $f_2(y' t)$. The value ξ, which corresponds y=0, we will designate through ξ_0. With these designations we will obtain for function F, determined by formula (2.22), the following approximation:

$$F = \frac{i}{2}\sqrt{-\frac{dy'}{d\xi'}}\sqrt{-\frac{dy}{d\xi}}\,\times$$

$$\times w_1(\xi')\left\{w_2(\xi) - \frac{w_2'(\xi_0) + q\left(\frac{dy}{d\xi}\right)_0 w_2(\xi_0)}{w_1'(\xi_0) + q\left(\frac{dy}{d\xi}\right)_0 w_1(\xi_0)}\,w_1(\xi)\right\}. \tag{3.08}$$

Being substituted into formula (2.23), this expression can be

used for calculating the field both in the shadow zone and in the illuminated region. In the shadow zone attenuation factor (and also function Ψ) is calculated from a number of the deductions, which correspond to the roots of the denominator

$$w_1'(\xi_0) - q\left(\frac{dy}{d\xi}\right)_0 w_1(\xi_0) = 0. \qquad (3.09)$$

In the illuminated region function Ψ is calculated directly with th aid of the contour integral, moreover the main section of integration will lie/rest near the real negative values of t. But with negative t of value ξ_0, ξ and ξ' they will be also negative. Assuming that these values are sufficiently great, it is possible to replace for function w_1 and w_2 with their asymptotic expressions

$$w_1(\xi) = e^{i\frac{\pi}{4}}(-\xi)^{-\frac{1}{4}}e^{i\frac{2}{3}(-\xi)^{3/2}}. \qquad (3.10)$$

$$w_2(\xi) = e^{-i\frac{\pi}{4}}(-\xi)^{-\frac{1}{4}}e^{-i\frac{2}{3}(-\xi)^{3/2}}. \qquad (3.11)$$

This replacement is reduced to the fact that for functions $f_1(y, t)$ and $f_2(y, t)$ are utilized asymptotic expressions (2.19) and (2.20).

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As a result for function F is obtained [according to the formula (2.22)] the following expression:

$$F = \frac{i}{2} \frac{1}{\sqrt[4]{p(y)-t}\,\sqrt[4]{p(y')-t}} \left\{ \exp\left[i \int_{y}^{y'} \sqrt{p(u)-t}\, du \right] - \right.$$

$$\left. - \frac{q - i\sqrt{p(0)-t}}{q - i\sqrt{p(0)-t}} \exp\left[i \int_{0}^{y} \sqrt{p(u)-t}\, du - i \int_{0}^{y'} \sqrt{p(u)-t}\, du \right] \right\}. \qquad (3.12)$$

This formula presents the generalization of formula (6.11) of Chapter 14. The latter is obtained from (3.12) after replacement of p(y) by zero.

The substitution of value (3.12) into the contour integral gives for the attenuation factor the expression, which consists of two members of whom the first corresponds to the incident wave, and the second - to wave, once reflected from the earth's surface with Fresnel's coefficient. The incident wave presents the imposition of waves with the phase

$$\omega(t) = xt - \int_{\xi}^{\xi'} \sqrt{p(u)-t}\, du, \qquad (3.13)$$

but the wave reflected presents the imposition of waves with the phase

$$\varphi(t) = xt - \int_0^{y} \sqrt{p(u)-t}\,du - \int_0^{y'} \sqrt{p(u)-t}\,du. \qquad (3.14)$$

These expressions correspond to geometric optic/optics. Integrals can be calculated according to the method of steady state, moreover the phase of the incident wave will be equal to outer limit $\omega(t)$, and the phase of the wave reflected - to outer limit $\varphi(t)$. Function $\omega(t)$ reaches its outer limit with t, determined from the equation

$$\omega'(t) \equiv x - \frac{1}{2}\int_0^{y} \frac{du}{\sqrt{p(u)-t}} = 0. \qquad (3.15)$$

while function $\varphi(t)$ - with t, determined from the equation

$$\varphi'(t) \equiv x - \frac{1}{2}\int_0^{y} \frac{du}{\sqrt{p(u)-t}} - \frac{1}{2}\int_0^{y'} \frac{du}{\sqrt{p(u)-t}} = 0. \quad (3.16)$$

From the point of view of geometric optic/optics the horizon distance is determined from the condition so that this point could reach the wave reflected with req. phase. Extreme value t, at which this still occurs, exists t=p(0). This value must be at the same time the root of equation (3.16).

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Consequently, between x, y and y' must exist the

relationship/ratio

$$x = \frac{1}{2}\int_{0}^{\xi}\frac{du}{\sqrt{p(u)-p(0)}} + \frac{1}{2}\int_{0}^{\xi_0}\frac{du}{\sqrt{p(u)-p(0)}}, \qquad (3.17)$$

which gives formula for the horizon distance during the normal refraction.

More exact expression (3.08) for F shows that at value of t=p(0) to apply formula (3.12) already is impossible. In fact, at this value of t value ξ_0 turns into zero, and to use formulas (3.10) and (3.11)it goes without saying inadmissibly. Nevertheless, it is possible to consider that value x, determined from (3.17), approximately gives the boundary, which is determining the illuminated region where is applicable reflecting formula, from the shadow region where let us use the series/row of deductions. As the in other words, it is possible to consider that the amplitude of field begins rapidly to decrease, when x, growing/rising, i passed through value (3.17). In this sense it is possible to apply term "horizon distance" and in the diffractive theory.

4. Asymptotic integration of differential equation with the coefficient, which has the minimum.

In the presence of superrefraction the given refractive index M(h) there will not be by monotonic function from the

height/altitude, but will have one or several minimums, which correspond to separate waveguide channels. We will examine the case of one minimum; th appropriate height/altitude we will call the height/altitude of inversion and designate through h_i.

Coefficient p(y) of the differential equation

$$\frac{d^2f}{dy^2} + p(y)f = tf \qquad (4.01)$$

is proportional to M(h) and therefore it will also have one minimum at value $y = y_i$, appropriate $h = h_i$.

We will count p(y) analytic function from y. Equation p(y)=t will have in the region interesting us two roots: $y=b_1$ and $y=b_2$.

With t real and lying between p(0) and $p(y_i)$ both roots will be real; at other values o t the roots can be composite.

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We should have this asymptotic expression for functions $f_1(y,t)$ and $f_2(y,t)$which would be correctly evenly at all values of y and t in question, switching on value $t = p(y_i)$, with which roots b_1 and b_2 coincide.

The approximation for f_1 and f_2 used in paragraph 2 through the Airy's functions here narrower is not applicable. Its validity was based on what with the aid of substitution (3.01)-(3.02), that defines ξ as holomorphic functio from y, equation (4.01) approximately was reduced to the equation

$$\frac{d^2w}{d\xi^2} - \xi w = 0, \qquad (4.02)$$

in which coefficient with the unknown function it had the same monotonic character as in the initial equation. Now we should take as the standard equation no longer equation (4.02) for the Airy's functions, but the equation

$$\frac{d^2g}{d\zeta^2} - \left(\frac{1}{4}\zeta^2 - \nu\right) g = 0 \qquad (4.03)$$

for the functions of parabolic cylinder, since this is the simple equation in which the coefficient with the unknown function has the same character (with one minimum)and coefficient p(y). Th substitution, which connects ζ and y, must be selected so that the value p(y)-t would become zero simultaneously with value $\frac{1}{4}\zeta^2 - \nu$ and so that at the high values of these values would be obtained correct asymptotic expressions. These conditions satisfies the substitution

$$\int_{y_1}^{y} \sqrt{p(y) - t}\, dy = \frac{1}{2} \int_{-2\sqrt{\nu}}^{\zeta} \sqrt{\zeta^2 - 4\nu}\, d\zeta, \qquad (4.04)$$

when parameter ν is selected so that it would be

$$\int_{b_1}^{b_2} \sqrt{p(y)-t}\, dy = \frac{1}{2} \int_{-2\sqrt{\nu}}^{2\sqrt{\nu}} \sqrt{\zeta^2 - 4\nu}\, d\zeta. \tag{4.05}$$

Integral in right side (4.05) is equal to

$$\frac{1}{2} \int_{-2\sqrt{\nu}}^{2\sqrt{\nu}} \sqrt{\zeta^2 - 4\nu}\, d\zeta = i\pi\nu. \tag{4.06}$$

Therefore equation (4.05) can be written in the form

$$i\pi\nu = \int_{b_1}^{b_2} \sqrt{p(y)-t}\, dy. \tag{4.07}$$

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It gives ν as function of t. This function will be holomorphic near $t = p(y_0)$, namely we will have

$$\nu = \frac{p(y_0)-t}{\sqrt{2p''(y_0)}} + \dots \tag{4.08}$$

Assuming/setting

$$S = \int_{c}^{y} \sqrt{p(y)-t}\, dy. \tag{4.09}$$

$$S_0 = \frac{1}{2} \int_{c}^{b_1} \sqrt{p(y)-t}\, dy + \frac{1}{2} \int_{c}^{b_2} \sqrt{p(y)-t}\, dy. \tag{4.10}$$

we can write substitution (4.04) in the form

$$S - S_0 = \frac{1}{2} \int_{0}^{\zeta} \sqrt{\zeta^2 - 4\nu}\, d\zeta. \tag{4.11}$$

the right side of this expression is equal to

$$\frac{1}{2} \int_{0}^{\zeta} \sqrt{\zeta^2 - 4\nu}\, d\zeta = \frac{1}{4} \zeta \sqrt{\zeta^2 - 4\nu} -$$

$$- \nu \lg \left(\zeta + \sqrt{\zeta^2 - 4\nu}\right) + \frac{\nu}{2} \lg 4\nu. \tag{4.12}$$

Hence we can conclude that with $\zeta>0$ value $S-S_0-\frac{\nu}{2}\lg\nu$ will be holomorphic function from ν near $\nu=0$. and with $\zeta<0$ holomorphic will be values $S-S_0-\frac{\nu}{2}\lg\nu$ and S. However, since with y=0 (on the earth's surface) knowingly $\zeta<0$, then we deal concerning the second case, and then holomorphic functionfrom ν will be sum $S_0+\frac{\nu}{2}\lg\nu$. This observation will be necessary by us subsequently.

In the asymptotic approximation of solution of equations (4.01) in question and (4.03) they are connected with the relationship/ratio

$$f=\sqrt{\frac{dy}{d\zeta}}\,g. \qquad (4.13)$$

The solutions of equation (4.03) are the functions which are expressed as the functions of parabolic cylinder $D_n(z)$, satisfying the equation

$$\frac{d^2D_n(z)}{dz^2}+\left(n+\frac{1}{2}-\frac{1}{4}z^2\right)D_n(z)=0. \qquad (4.14)$$

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Functions $D_n(z)$ are well investigated. We will not enumerate their

properties, but let us refer to the book of Whittaker and Watson the "Course of contemporary analysis", Vol. 2, GITTL, 1934, where are given the principal formulas. As determination $D_n(z)$ it is possible to take the series/row

$$D_n(z) = \frac{2^{-\frac{n}{2}-1}}{\Gamma(-n)} e^{-\frac{z^2}{4}} \sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{m-n}{2}\right)}{\Gamma(m+1)} 2^{\frac{m}{2}} (-z)^m. \quad (4.15)$$

Equation (4.03) is obtained from (4.14) by replacement z on $\zeta e^{-i\frac{\pi}{4}}$ and n+1/2 on $i\nu$. The solutions of equation (4.03) will be the functions

$$g_1(\zeta) = D_{i\nu-\frac{1}{2}}\left(e^{-i\frac{\pi}{4}}\zeta\right), \quad (4.16)$$

$$g_2(\zeta) = D_{-i\nu-\frac{1}{2}}\left(e^{i\frac{\pi}{4}}\zeta\right). \quad (4.17)$$

With real ones ν and ζ values $g_1(\zeta)$ and $g_2(\zeta)$ will be composite conjugated/combined.

From properties $D_n(z)$ it escape/ensue

$$g_1(-\zeta) = e^{-\nu\pi - i\frac{\pi}{2}} g_1(\zeta) + \frac{\sqrt{2\pi}}{\Gamma\left(\frac{1}{2}-i\nu\right)} e^{-\frac{\nu\pi}{4}+i\frac{\pi}{4}} g_2(\zeta), \quad (4.18)$$

and also

$$g_2(-\zeta) = e^{-\nu\pi - i\frac{\pi}{2}} g_2(\zeta) - \frac{\sqrt{2\pi}}{\Gamma\left(\frac{1}{2}+i\nu\right)} e^{-\frac{\nu\pi}{4}-i\frac{\pi}{4}} g_1(\zeta). \quad (4.19)$$

For us are essential asymptotic expressions for $g_1(\zeta)$ and $g_2(\zeta)$. In the region, which adjoins the positive real axis, we have

$$g_1(\zeta) = e^{\frac{\pi\nu}{4} - i\frac{\pi}{8}} e^{i\frac{\zeta^2}{4}} \zeta^{i\nu - \frac{1}{2}} \left(1 - \frac{i\nu^2 - 2\nu - i\frac{3}{4}}{2\zeta^2} + \ldots\right). \tag{4.20}$$

Using formula (4.12), we can also write

$$g_1(\zeta) = e^{\frac{\pi\nu}{4} + i\frac{\pi}{8}} e^{-i\frac{\nu}{2} + i\frac{\nu}{2}\lg\nu} \times$$
$$\times \frac{1}{\sqrt[4]{\zeta^2 + 4\nu}} \exp\left[\frac{i}{2}\int_0^{\zeta} \sqrt{\zeta^2 + 4\nu}\, d\zeta\right]. \tag{4.21}$$

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Latter/last expression correctly also with large ones ν. Asymptotic expression for $g_2(\zeta)$ is obtained hence by replacement of i on -i.

In order to obtain the formula, valid near the negative real axis, we they must use relationship/ratio (4.18).

We will have

$$g_1(\zeta) = e^{-\frac{3\nu\pi}{4} - i\frac{3\pi}{8}} e^{-i\frac{\nu}{2} - \frac{i\nu}{2}\lg\nu} \times$$

$$\times \frac{1}{\sqrt[4]{\zeta^2 - 4\nu}} \exp\left[-\frac{i}{2}\int_0^{\zeta}\sqrt{\zeta^2 - 4\nu}\,d\zeta\right] +$$

$$+ \frac{\sqrt{2\pi}}{\Gamma\left(\frac{1}{2} - i\nu\right)} e^{-\frac{\nu\pi}{4} + i\frac{\pi}{8}} e^{i\frac{\nu}{2} - \frac{i\nu}{2}\lg\nu}$$

$$\times \frac{1}{\sqrt[4]{\zeta^2 - 4\nu}} \exp\left[-\frac{i}{2}\int_0^{\zeta}\sqrt{\zeta^2 - 4\nu}\,d\zeta\right]. \qquad (4.22)$$

Now we can construct the solution of equation (4.01), which satisfies all stated requirements.

Let us assume

$$c_1(\nu) = e^{i\frac{\pi}{8} - \frac{\pi\nu}{4}} e^{i\left(\frac{\nu}{2} - \frac{\nu}{2}\lg\nu - S_0\right)}. \qquad (4.23)$$

In view of the property of value S_0 noted above, index in (4.23) is holomorphic function from ν also near $\nu = 0$.

The proper solution of equation for the high-altitude factor will be the function

$$f_1(y, t) = c_1(\nu)\sqrt{2\frac{dy}{d\zeta}}\, g_1(\zeta). \qquad (4.24)$$

Higher than the layer of inversion (with $S-S_0 \gg 1$) this function has the asymptotic expression

$$f_1(y, t) = \frac{e^{i\frac{\pi}{4}}}{\sqrt[4]{p(y) - t}} e^{iS - 2iS_0}, \qquad (4.25)$$

ensuing/escaping/flowin out from (4.21).

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Lower than the layer of inversion (with S_0-S>>1) asymptotic expression for $f_1(y,t)$ will have the form

$$f_1(y,t) = \chi_1(\nu) \frac{e^{i\frac{\pi}{4}}}{\sqrt[4]{p(y)-t}} e^{iS-2iS_0} + e^{-\nu\pi} \frac{e^{-i\frac{\pi}{4}}}{\sqrt[4]{p(y)-t}} e^{-iS}; \quad (4.26)$$

where we placed

$$\chi_1(\nu) = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{1}{2} - i\nu\right)} e^{-\frac{\nu\pi}{2}} e^{i(\nu - \nu \lg \nu)}. \qquad (4.27)$$

With the aid of the known asymptotic expression for function $\Gamma(1/2 - i\nu)$ it is easy to show that at high positive values ν function $\chi_1(\nu)$ approaches unity. Since when $\nu \gg 1$ the second term in (4.26) becomes small in comparison with the first, both expressions for $f_1(y,t)$ will then in form coincide It is substantial, however, then that our expressions for $f_1(y,t)$ are valid not only with the large ones, but also at low values ν, up to $\nu = 0$, and that they represent holomorphic functions from ν near $\nu = 0$.

The corresponding expressions for $f_2(y,t)$ will be obtained from preceding/previous by replacement of i on -i. In order to extract

them clearly, let us assume

$$c_2(\nu) = e^{-i\frac{\pi}{8} - \frac{\pi\nu}{4}} e^{-i\left(\frac{\nu}{2} - \frac{\nu}{2}\lg\nu - S_0\right)}. \qquad (4.28)$$

$$\chi_2(\nu) = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{1}{2} - i\nu\right)} e^{-\frac{\nu\pi}{2}} e^{-i(\nu - \nu\lg\nu)}. \qquad (4.29)$$

Then it will be

$$f_2(y,t) = c_2(\nu)\sqrt{2\frac{d\nu}{d\zeta}}\, g_2(\zeta). \qquad (4.30)$$

and asymptotic expressions for $f_2(y,t)$ will take the following form:

with $S-S_0>>1$

$$f_2(y,t) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt[4]{p(y)-t}} e^{-iS+iS_0}, \qquad (4.31)$$

with $S_0-S>>1$

$$f_2(y,t) = \chi_2(\nu)\frac{e^{-i\frac{\pi}{4}}}{\sqrt[4]{p(y)-t}} e^{-iS+iS_0} -$$

$$- e^{-\nu\pi}\frac{e^{i\frac{\pi}{4}}}{\sqrt[4]{p(y)-t}} e^{iS}. \qquad (4.32)$$

Thus, the problem of the asymptotic integration of equation for the high-altitude factors we have solved.

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5. Investigation of attenuation factor.

To us is to now substitute those found for $f_1(y,t)$ and $f_2(y,t)$ of expression in formula (2.22) for F and to investigate attenuation factor V or connected with it function Ψ. For simplicity we will be restricted to case $q=\infty$, which corresponds to horizontal polarization In this case function F is reduced to the form

$$F(t,y,y',\infty)=\frac{1}{D_{12}(t)}f_1(y',t)\left[f_2(y,t)-\frac{f_2(0,t)}{f_1(0,t)}f_1(y,t)\right]. \quad (5.01)$$

For functions (4.24) and (4.30) the Wronskian D_{12} is equal to the constant value

$$D_{12}=-2i. \qquad (5.02)$$

which more easily in all is derived/concluded from asymptotic expressions (4.25) and (4.21). We will assume that $y'>y_i$, so that $S(y')-S_0>>1$, and let us examine two cases: when the second height/altitude is also higher and when it is lower than the layer of inversion. In the first case we will consider $S(y)-S_0>>1$ that makes it possible to us expressions (4.25) and (4.31) for f_1 and f_2. In the second case we will count $S_0-S(y)>>1$ and use expressions (4.26) and (4.32).

In the first case we will have

$$F=\frac{i}{2}\,\frac{e^{iS(y')-2iS_0}}{\sqrt[4]{\varepsilon(y')-t}\,\sqrt[4]{\varepsilon(y)-t}}\cdot$$

$$\cdot\left[e^{-iS-2iS_0}-\frac{e^{-\pi v}-i\chi_2 e^{2iS_0}}{e^{-\pi v}-i\chi_1 e^{-2iS_0}}e^{iS-2iS_0}\right]. \qquad (5.03)$$

The individual members of this expression allow/assume interpretation on the basis of geometric optic/optics. It is obvious that the wave, which goes downward, must have phase factor e^{-iS}, and the wave, which goes from bottom to top - phase factor e^{iS}. Expression (5.03) shows that there is only one wave, whic goes downward, namely incident wave with the complete phase

$$\varphi(t) = xt - S(y') - S(y) \qquad (5.04)$$

[they added here member xt from the exponential factor in the integral (2.23)]. This phase coincides with phase (3.13) of the case of normal refraction, which is natural, since this wave yet did not reach the layer of inversion.

As far as the waves, which go from bottom to top, are concerned, them there will be countless set; these waves will be obtained by the resolution of second term (5.03) in the series/ro according to degrees e^{-2v}. They will correspond to the waves, repeatedly reflected from the earth's surface and from the layer of inversion.

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The phase of the wave, once reflected from the earth's surface, will be equal to

$$\varphi(t) = xt + S(y') - S(y) + \operatorname{arc}\frac{\chi_2}{\chi_1}. \qquad (5.05)$$

This expression differs from (3.14) in terms of the latter/last term which cannot be obtained from the geometric optic/optics. This term is equal to

$$\arg\frac{\chi_2}{\chi_1} = \arg\frac{\Gamma\left(\frac{1}{2} - i\nu\right)}{\Gamma\left(\frac{1}{2} + i\nu\right)} + 2\nu\lg\nu - 2\nu. \qquad (5.06)$$

With large positive ones ν it becomes zero, but with small ones ν it play important role, since because of it entire/all phase $\varphi(t)$ remains holomorphic function from ν near $\nu = 0$, in other words, near $t = p(y)$.

Let us examine now the case when point y is located below the layer of inversion, moreover $S_c - S \gg 1$.

Using expressions (4.26) and (4.32) and utilizing the equality

$$\chi_1(\nu)\chi_2(\nu) - e^{-2\pi\nu} = 1, \qquad (5.07)$$

we obtain after some linings/calculations

$$F = \frac{e^{iS(y') - 2iS_0}}{\sqrt[4]{p(y') - t}\sqrt[4]{p(y) - t}} \frac{\sin S(y)}{\chi_1 e^{-2iS_0} - ie^{-\pi\nu}}. \qquad (5.08)$$

In this case there is not by one, but a countless quantity of waves, which go downward, since to the incident wave ar connected the

waves, reflected from the layer of inversion as from upper boundary. Furthermore, is an infinite quantity of waves, reflected from the earth/ground and which go from bottom to top. All these waves are formally obtained by expansion (5.08) into the geometric progression according to degrees e^{-2iv}.

The complete phase of the wave, which did not undergo terrain echo, is equal to

$$\omega(t) = xt - S(y') - S(y) - \operatorname{arc} \chi_1 \tag{5.09}$$

or

$$\omega(t) = xt - S(y') - S(y) + \frac{1}{2} \operatorname{arc} \frac{\chi_2}{\chi_1}, \tag{5.10}$$

while the complete phase of the once reflected wave is equal to

$$\varphi(t) = xt - S(y') - S(y) + \frac{1}{2} \operatorname{arc} \frac{\chi_2}{\chi_1}. \tag{5.11}$$

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Expression for $\omega(t)$ does not coincide with (3.13) or (5.04), which is natural, since the incident wave traversed the layer of inversion. Expression (5.11) differs from (5.05) in terms of the fact that the additional term enters with factor 1/2.

We spoke, until now, about the phases of different members of integrand. To each such term corresponds in the attenuation factor integral on t. If we calculate these integrals according to the method of steady state, then each of the gives in the attenuation factor the term, which presents wave with the phase, equal to the outer limit of the phase of integrand.

Of course this method of calculating the attenuation factor is applicable only in the illuminated region, whereas in the region of shadow it is necessary to use the series/row of deductions.

6. Formula for the horizon distance.

During the normal refraction (paragraph 2) we estimated distance of the horizon/level as this value of the horizonta distance of x which gives the boundary between the region of the applicability of reflecting formula and the region of the applicability of the series/row of deductions. At this value of x the extremum of the phase of the wave reflected falls at extreme value of t at which phase itself is still real.

In the presence of superrefraction are many waves reflected. But we can expect that main role plays waves, once reflected from the earth's surface. Since the "horizon distance is not the strictly

defined concept, we are right to refine it, understanding under it horizon distance for the once reflected wave.

The phases of the once reflected wave have found we in paragraph 5. According to (5.05) and (5.11) we have when $y' > y_i$, $y > y_i$.

$$q(t) = xt - \int_0^{y'} \sqrt{p(u) - t}\, du - \int_0^{y} \sqrt{p(u) - t}\, du - \operatorname{arc} \frac{\chi_2}{\chi_1} \quad (6.01)$$

and when $y' > y_i$, $y < y_i$

$$q(t) = xt - \int_0^{y'} \sqrt{p(u) - t}\, du - \int_0^{y} \sqrt{p(u) - t}\, du - \frac{1}{2} \operatorname{arc} \frac{\chi_2}{\chi_1}. \quad (6.02)$$

These formulas can be combined, after assuming

$$S^*(y, t) = \int_0^{y} \sqrt{p(u) - t}\, du + \frac{1}{2} \operatorname{arc} \frac{\chi_2}{\chi_1} \qquad (y > y_i), \quad (6.03)$$

$$S^*(y, t) = \int_0^{y} \sqrt{p(u) - t}\, du \qquad (y < y_i). \quad (6.04)$$

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Then it will be both when $y > y_i$, and when $y < y_i$

$$q(t) = xt - S^*(y', t) - S^*(y, t). \qquad (6.05)$$

Let us note that S* is holomorphic function from t near $t = p(y_i)$.

By discussin as in paragraph 2, we will obtain for the horizon distance the expression

$$x = -\left[\frac{\partial S^*(y,t)}{\partial t} + \frac{\partial S^*(y,t)}{\partial t}\right]_{t=p(y_i)}. \qquad (6.06)$$

Let us write this expression in the more explicit form. According to (4.08) near $t = p(y_i)$ we have

$$\nu = \frac{p(y_i) - t}{\sqrt{2p''(y_i)}}. \qquad (6.07)$$

On the other hand, near $\nu = 0$

$$\frac{1}{2}\operatorname{arc}\frac{\Gamma\left(\frac{1}{2} - i\nu\right)}{\Gamma\left(\frac{1}{2} - i\nu\right)} = (C - 2\lg 2)\nu - \cdots \qquad (6.08)$$

and, therefore,

$$\frac{1}{2}\operatorname{arc}\frac{\chi_2}{\chi_1} = \nu(C - 1 - \lg 4\nu) + \cdots, \qquad (6.09)$$

where C=0.577... - Euler's constant. Therefore when $y > y_i$

$$-\frac{\partial S^*}{\partial t} = \frac{1}{2}\int_0^y \frac{du}{\sqrt{p(u) - t}} + \frac{1}{\sqrt{2p''(y_i)}}(C - \lg 4\nu). \qquad (6.10)$$

This expression has a limit when $t \to p(y_i)$, $\nu \to 0$. When $y < y_i$ latter/last term is absent, and into the integral it is possible to directly substitute value $t = p(y_i)$. Therefore during $y < y_i$ Budde's lifetime

$$-\frac{\partial S^*}{\partial t} = \frac{1}{2}\int_0^y \frac{du}{\sqrt{p(u) - p(y_i)}}. \qquad (6.11)$$

The presence of the second term in formula (6.02) causes the dependence of horizon distance on the wavelength. In order to come to

light/detect/expose this dependence, let us return from reduced coordinates x, y to the usual coordinates s, h, where s - horizontal distance and h - height/altitude. Designating through $\mu(h)$ the given refractive index without factor 10^6, we will have

$$p(y) = 2m^2\mu(h), \tag{6.12}$$

where m is value (2.04).

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Instead of t let us introduce the parameter τ, connected with t with the relationship/ratio

$$t = 2m^2\tau. \tag{6.13}$$

Then

$$\int_0^y \sqrt{p(u) - t}\, du = k \int_0^h \sqrt{2\mu(h) - 2\tau}\, dh, \tag{6.14}$$

$$xt = ks\tau. \tag{6.15}$$

Value ν will be now approximately equal to

$$\nu = \frac{k}{\sqrt{\mu'(h_1)}} [\mu(h_1) - \tau]. \tag{6.16}$$

Formula for the horizon distance will be obtained from the condition

$$\frac{1}{k}\frac{\partial q}{\partial \tau} = 0 \quad (\text{при}^{(1)}\ \tau = \mu(h_i)). \tag{6.17}$$

Key: (1). with.

where the phase φ is assumed to be that expressed through the new values.

Let us assume

$$F(h) = \int_0^h \frac{dh}{\sqrt{2\mu(h) - 2\mu(h_i)}} \quad (\text{при}^{(1)}\ h < h_i), \tag{6.18}$$

$$F(h) = \lim_{\tau \to \mu(h_i)} \left\{ \int_0^h \frac{dh}{\sqrt{2\mu(h) - 2\tau}} + \right.$$

$$\left. - \frac{1}{\sqrt{\mu''(h_i)}} \left[C - \lg \frac{4k(\mu(h_i) - \tau)}{\sqrt{\mu''(h_i)}} \right] \right\} \quad (\text{при}^{(1)}\ h > h_i). \tag{6.19}$$

Key: (1). with.

Then the obtained from condition (6.17) formula for the horizon distance is written

$$s = F(h') + F(h). \tag{6.20}$$

Let us compare the values of horizon distance for th identical heights/altitudes, but for the different wavelengths. Wavelength enters into expression for F(h) only when $h > h_i$ and only into the log term. Let with $\lambda=\lambda_1=2\pi/k_1$ the horizon distance be equal to s_1, and with $\lambda=\lambda_2=2\pi/k_2$ it is equal to s_2.

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Composing a difference in expressions ·0), we obtain when $h > h_i$

$$s_2 - s_1 = \frac{2}{\sqrt{\mu''(h_i)}} \lg \frac{k_2}{k_1} = \frac{2}{\sqrt{\mu''(\cdot\ )}} \lg \frac{\lambda_1}{\lambda_2} \qquad (6.21)$$

and when $h < h_i$

$$s_2 - s_1 = \frac{1}{\sqrt{\mu''(h_i)}} \lg \frac{k_2}{k_1} = \frac{1}{\sqrt{\mu''(h_i)}} \lg \frac{\lambda_1}{\lambda_2}. \qquad (6.22)$$

This difference depends, besides the relation of wavelengths, only on the course of the given refractive index near its minimum

Let us apply our general formulas to the case when the given refractive index μ(h) depends on height/altitude according to the hyperbolic law

$$\mu(h) = \mu(h_i) + \frac{1}{a} \frac{(h - h_i)^2}{h - l}, \qquad (6.23)$$

where a - radius of terrestial globe; l — parameter.

In this case

$$\mu''(h_i) = \frac{2}{a(h_i - l)}. \qquad (6.24)$$

The integrals, entering in φ(t), will be elliptical, but when $\tau = \mu(h_i)$ they are calculated elementarily, and we obtain for F(h) the following expressions. When $h < h_i$.

$$F(h) = -\sqrt{2a(h-l)} - \sqrt{2al} +$$
$$-\sqrt{\frac{a(h_i-l)}{2}}\left\{\lg\frac{\sqrt{h_i-l}-\sqrt{h-l}}{\sqrt{h_i-l}-\sqrt{h+l}} - \lg\frac{\sqrt{h_i+l}+\sqrt{l}}{\sqrt{h_i+l}-\sqrt{l}}\right\} \quad (6.25)$$

and when $h > h_i$

$$F(h) = \sqrt{2a(h-l)} + \sqrt{2al} -$$
$$-\sqrt{\frac{a(h_i-l)}{2}}\left\{\lg\frac{\sqrt{h-l}-\sqrt{h_i-l}}{\sqrt{h+l}-\sqrt{h_i+l}} + \lg\frac{\sqrt{h_i-l}+\sqrt{l}}{\sqrt{h_i+l}-\sqrt{l}}\right\} + \Delta s, \quad (6.26)$$

where

$$\Delta s = \sqrt{\frac{a(h_i-l)}{2}}\left\{C_1 - \frac{1}{2}\lg\frac{2k^2(h_i-l)^3}{a}\right\}. \quad (6.27)$$

Here

$$C_1 = 7\lg 2 - 4 + C = 1{,}429. \quad (6.28)$$

For the comparison let us not that the horizon distance in the absence of refraction is equal, as is known,

$$s' = \sqrt{2ah'} + \sqrt{2ah}. \quad (6.29)$$

Thus, an increase in the horizon distance as a result of the refraction is equal

$$s - s' = \left[F(h') - \sqrt{2ah'}\right] + \left[F(h) - \sqrt{2ah}\right]. \quad (6.30)$$

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In all preceding/previous reasonings we assumed that heights/altitudes h and h' were small in comparison with a radius of Earth a. But the preceding/previous formulas are applicable also to the case of the wave, which goes from infinity (for example, from the Sun). A difference $F(h') - \sqrt{2ah'}$ has when $h' \to \infty$ the final limit, equal to

$$\lim_{h' \to \infty} [F(h') - \sqrt{2ah'}] =$$

$$= \sqrt{2al} - \sqrt{\frac{a(h_i - l)}{2}} \lg \frac{\sqrt{h_i - l} + \sqrt{l}}{\sqrt{h_i - l} - \sqrt{l}} - \Delta s. \qquad (6.31)$$

Replacing in (6.30) the first two members by their limiting value, we will obtain for an increase in the horizon distance the following expressions:

when $h < h_i$

$$s - s' = 2\sqrt{2al} - \sqrt{2a(h-l)} - \sqrt{2ah} - \Delta s -$$

$$- \sqrt{\frac{a(h_i - l)}{2}} \left\{ \lg \frac{\sqrt{h_i - l} + \sqrt{h - l}}{\sqrt{h_i - l} - \sqrt{h - l}} - 2 \lg \frac{\sqrt{h_i - l} + \sqrt{l}}{\sqrt{h_i - l} - \sqrt{l}} \right\} \qquad (6.32)$$

and when $h > h_i$

$$s - s' = 2\sqrt{2al} - \sqrt{2a(h-l)} - \sqrt{2ah} - 2\Delta s -$$

$$- \sqrt{\frac{a(h_i - l)}{2}} \left\{ \lg \frac{\sqrt{h - l} + \sqrt{h_i - l}}{\sqrt{h - l} - \sqrt{h_i - l}} - 2 \lg \frac{\sqrt{h_i - l} + \sqrt{l}}{\sqrt{h_i - l} - \sqrt{l}} \right\}. \qquad (6.33)$$

To this increase in the distance corresponds "lead angle"

$$\delta = \frac{s - s'}{a}. \qquad (6.34)$$

Since present theory does not consider refraction in the upper levels of atmosphere, then for the comparison with the observed lead angle it is necessary to value (6.34) to add the value o normal refraction at the horizon/level.

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Chapter 16.

The radiowave propagation near the horizon/level during the superrefraction [1].

FOOTNOTE [1]. Foch, Weinstein Belkin, 1956. ENDFOOTNOTE.

In the beginning of chapter is more precisely formulated the concept of the horizon/level in the presence of surface tropospheric waveguide. Further are investigated approximations (of type of reflecting formula) for the attenuation factor. Under the assumption of the hyperboli law of a change of the refractive index with the height/altitude are derived the formulas for the horizon distance of the straight/direct and reflected waves. In second half of chapter are given numerical results for several typical examples, moreover it is assumed that the transmitting antenna is arranged/located highly above the layer of inversion, and receiving - within the layer of inversion on the small height/altitude. The obtained results give the estimation of the possible values of attenuation factor at the horizons/levels and is shown its dependence on the distance and on the wavelength. The conducted investigation shows the advisability of

the introduction of the concept of the horizons/levels during the analysis of hyperdistant propagation.

1. Introduction.

The theory of radiowave propagation above the spherical earth's surface in the presence of the unhomogeneous atmosphere the refractive index of which depends only on height/altitude was developed in Chapters 14 and 15. In Chapter 15 is given investigation of attenuation factor in the unhomogeneous atmosphere near the horizon, moreover the concept of the horizon/level is determined for the heterogeneous (laminar) atmosphere of any type. The determination of the horizon/level introduced there coincides, in the case of unhomogeneous atmosphere without the inversion, with the determination of the boundary of shadow, which escape/ensues from the laws of geometric optic/optics. But if is an inversion of the given refractive index, then the horizon/level is necessary to find of the thinner wave considerations: its position in this case depends also on wavelength.

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Assuming that during the removal/distance beyond the horizon/level the attenuation factor rapidly decreases, then it is

possible to conditionally consider (as this is made in Chapter 15) that the horizon distance estimates distance radiowave propagation. In this wa is obtained simple formula for the superstandard range of radio waves during the superrefraction; into this formula enter the height of receiving and transmitting antennas, the wavelength and the parameters, which characterize the course of a change in the given refractive index with the height/altitude (M - profile/airfoil). Especially simple form takes the formula of distance for the given refractive index, which depends on the height/altitude according to the hyperbolic law (paragraph 5 Chapter 15).

The analysis of hyperdistant propagation, produced in Chapter 15 on the basis of the concept of the horizon/level, requires, however, some refinements. It is first of all desirable to explain, what values accepts attenuation factor at the horizon/level and as depends attenuation factor near the horizon/level on distance, the wavelength and parameters of the layer of inversion (height/altitude of this layer, its average/mean gradient, etc.). For this it is obvious it is necessary to calculate attenuation factor in some special cases, since in general form this problem to solution does not yield. If we in this case will explain how rapidly attenuation factor decreases in the shadow region (beyond the horizon/level) and as rapidly it grows/rises to the values of the order of one during the removal/distance from the horizon/level into the illuminated region,

then we thereby is checked, in what measure the horizon/level determines in the practical cases the superstandard range of radio waves.

In view of the large labor expense for the calculations of attenuation factor during the superrefraction it is possible to conduct calculations only for a small number of typical cases. No comprehensive calculations as during the normal radiowave propagation, cannot be here fulfilled. Therefore we were restricted to the calculation of attenuation factor as the functions of dimensionless coordinate ζ in four cases, which allow for the fixed/recorded M-curved and at the fixed/recorded heights/altitudes of the corresponding points to construct the dependence of attenuation factor on the horizontal distance between these points for four wavelengths, which relate as by 1:3:9:27 (cf. paragraph 7).

In this way it proves to be possible to refine the sense of the concepts of the horizon distance and superstandard range and to answer the series/row of the questions presented above, in particular by a question about the dependence of the phenomenon of hyperdistant propagation on the wavelength.

Let us recall that the analysis of anomalous propagation, given in Chapter 15, let us use if and only if one of the corresponding

points is located above the lowest layer of the atmosphere of inversion, another point can be both within this layer and above it. Therefore during calculations of attenuation factor we were restricted to the case when one point is located highly above the layer of inversion and another - within the layer at the height equal to one fifth spot height of inversion.

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2. On the concept of the horizon/level in the presence of surface tropospheric waveguide.

Let us examine in more detail the concept of the horizon/level in the presence of surface waveguide (layer of inversion).

Let us recall, first of all, the beam/radiation treatment of normal and anomalous propagation.

For homogeneous atmosphere the given refractive index is a linear function of height/altitude. on plane s,h(s - distance on the earth/ground, h - height/altitude), the rays/beams, which emerge from source Q, take the form of the curves, turned by convexity to axis s(Fig. 1A). Horizon/level OO' is determined by ray/beam QOO', which concerns the earth's surface at point O. To the right of line of

horizon OO' is located shadow zone, where field penetrates only as a result of diffraction, to the left - illuminated region. For observation points which are located in the illuminated region (it is more left horizon/level OO'), is approximately applicable the reflecting formula, according to which the field is obtained as a result of the interference of the straight/direct ray/beam QP with earth-reflected wave QP'P.

Rays/beams ZhT from source Q, arranged/located within surface radio duct of height/altitude h_t (Fig. 1b), have within the waveguide a convexity upward (from axis s), and higher than the waveguide - convexity dow (as in Fig. 1A). Therefore ray/beam Q1 is passed into the space above the waveguide, and ray/beam Q2 proves to be "closed" within the waveguide. These two kinds of rays/beams are divided with the maximum ray/beam QO, which asymptotically approaches when $s \to \infty$ height/altitude $h = h_t$.

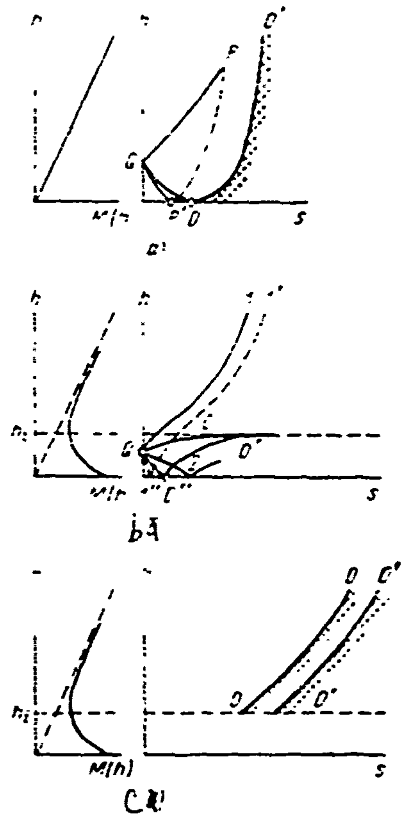

Fig. 1. Illustration to concept of horizon/level: a) during normal refraction; b) during superrefraction - on geometric optic/optics; c) during superrefraction on physical optics.

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Beside the straight/direct rays/beams into the space above the layer of inversion fall earth-reflected wave, for example Q1 "1', separated from the closed rays/beams by another maximum ray/beam QO" O',

asymptotically approaching height/altitude h_1 after single terrain echo. Closed prove to be all rays/beams, which emerge from the source within angle of OQO", formed by both maximum rays/beams.

The laws of geometric optic/optics in this example lead to the conclusion about the absenc of the horizon/level both within the waveguide and above. Actually/really, through observation points, arranged/located of the waveguide more to the right of rays/beams 1 and 1', pass the straight/direct rays/beams, which emerge from Q within the angle 1QO, and the reflected beams, which emerge within angle of 1 "QO". They thread entire space above the waveguide more to the right of rays/beams 1 and 1', and therefore geometrical shadow and, consequently, also the horizon/level, they are absent.

It is easy to see, however, that to the maximum rays/beams QO and QO" O' and to the rays/beams, close to the maximum ones, the laws of geometric optic/optics are not applied. From preceding/previous it is clear that precisely these almost maximum rays/beams would transfer (according to the laws of geometric optic/optics) electromagnetic energy to the large distances above the waveguide. Hence it follows that for solving the ques ion about the horizon/level and superstandard range during the superrefraction it is necessary to draw wave considerations.

This was made in Chapter 15, where it is shown that in the space above the waveguide is certain boundary O'O' (Fig. 1c), more to the right by which earth-reflected wave penetrate cannot. This boundary O'O' is the horizon/level when the layer of inversion is present, since more to the right this boundary, i.e., into the shadow region, the field (as in Fig. 1a) it can penetrate only by diffraction.

Together with boundary of O'O' is another boundary OO, more to the right by which cannot penetrate the straight/direct rays/beams, which did not experience terrain echo. Boundary O'O' is located more to the right boundary OO, since with the reflection from earth ray/beam proves to be more to the right parallel to it straight/direct ray/beam (cf. rays/beams Q1 and Q1" 1' in Fig. 1b). In band OO-O'-O' straight/direct rays/beams do not pass; therefore complete field to this band to beam/radiation treatment is not subordinated. More left boundary of OO compl-te electromagnetic field is obtained by the imposition of the straight/direct and reflected beam.

In view of this value of boundary of OO - limit of the applicability of reflecting formula - for it it is expedient to introduce the special name: we it will call the horizon/level of direct waves. In contrast to it boundary O'O' we will name/call the horizon/level of the waves reflected. While durin the normal

propagation these "norizons/levels" they coincide, in the case of their anomalous propagation it is necessary to distinguish. Horizons/levels O'O' and OO in Fig. 1c replace in the wave picture the maximum rays/beams QO" O' and QO (Fig. 1b), obtained from the geometric optic/optics.

These general/common/total considerations will be refined in paragraph 4.

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3. Basic formulas.

Attenuation factor V in the unhomogeneous atmosphere index of refraction of which depends only on height/altitude, can be represented in the form of the contour integral

$$V(x, y', y) = e^{-i\frac{\pi}{4}} \sqrt{\frac{\lambda}{\pi}} \int_C e^{ixt} F(t, y', y)\, dt. \qquad (3.01)$$

When the lowest layer of the atmosphere of inversion is present, if one of the corresponding points is located above the layer, and another - within it, for integrand F it is possible to take the following approximation [see the formula (5.08) of Chapter 15]:

$$F(t, y', y) = \frac{e^{i[S(y')-2S_0]}}{\sqrt{p(y')-t}\,\sqrt{p(y)-t}} \cdot \frac{\sin S(y)}{[\chi(t)\, e^{-2iS_0} - i e^{-\pi i/3}]}. \qquad (3.02)$$

Here y' and y - dimensionless heights/altitudes of source and observation point (y'>y, moreover $y' > y_i$, and $y < y_i$, where y_i — dimensionless spot height of inversion), x - the dimensionless horizontal distance between source and observation point, and p(y) - the function, connected with the given refractive index M(h) with the formula

$$p(y) = \frac{2m^2}{10^6} M(h) = 2m^2\left(n - 1 - \frac{h}{a}\right), \quad m = \left(\frac{ka}{2}\right)^{\frac{1}{3}}, \quad (3.03)$$

moreover n is a refractive index of air, a - radius of the Earth.

We assume that function M(h) takes the same form as in Fig. 1b and 1c. Therefore with this t the equation

$$p(y) - t = 0 \qquad (3.04)$$

has two roots of y_1 and y_2. When $p(y_i) < t < p(0)$ these roots are real and positive, when $t < p(y_i)$ they compositely conjugated/combined; when $t = p(y_i)$ they pour, and then $y_1 = y_2 = y_i$. Besides these two roots there can be, generally speaking, and other roots (negative or composite), but they a value do not have.

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Values S(y), S(y') and S_0 are given by the formulas

$$\left.\begin{aligned} S(y) &= \int_0^y \sqrt{p(y)-t}\,dy, \quad S(y') = \int_0^{y'} \sqrt{p(y)-t}\,dy, \\ S_0 &= \frac{1}{2}\int_0^{b_1} \sqrt{p(y)-t}\,dy + \frac{1}{2}\int_0^{b_2} \sqrt{p(y)-t}\,dy, \end{aligned}\right\} \qquad (3.05)$$

moreover for $t<p(y_i)$ radicals $\sqrt{p(y)-t}$ with real positive y must be taken in the arithmetical sense. For calculation S_0 when $t<p(y_i)$ it is necessary to analytically continue radical $\sqrt{p(y)-t}$ into the region composit y. We will consider that p(y) is analytic function [cf. below formula (3.18)], that allows/assumes this continuation.

Value ν is determined by the formula

$$\nu = \frac{1}{i\pi}\int_{b_1}^{b_2} \sqrt{p(y)-t}\,dy. \qquad (3.06)$$

For the real values of t value ν is also real, moreover sign ν is selected from the following considerations. When $y \approx y_i$ function p(y) can be replaced by the first members of Taylor series

$$p(y) = p(y_i) - \frac{1}{2}p''(y_i)(y-y_i)^2, \quad p''(y_i)>0.$$

after which integral (3.06) can be calculated, and we obtain for $t \approx p(y_i)$ the approximation formula

$$\nu = \frac{p(y_i)-t}{\sqrt{2p''(y_i)}}, \qquad (3.07)$$

in accordancewith which we we count $\nu>0$ when $t<p(y_i)$ and $\nu<0$ when $t>p(y_i)$; when $p(y_i)<t<p(0)$ formula (3.06) is deciphered

as follows:

$$\nu = -\frac{1}{\pi}\int_{y_1}^{y_2}\sqrt{t-p(y)}\,dy, \qquad (3.08)$$

where $\sqrt{t-p(y)}>0$, and $y_1<y_2$.

Function $\chi(\nu)$ is determined by the formula

$$\chi(\nu)=\frac{\sqrt{2\pi}\,e^{-\frac{\pi}{2}\nu+i(\nu-\nu\lg\nu)}}{\Gamma\left(\frac{1}{2}-i\nu\right)}, \qquad (3.09)$$

moreover when $\nu>0\ (t<p(y_1))$ for lg ν is taken principal value. In this case

$$\chi(\nu)\to 1 \text{ при}^{(1)}\ \nu\to\infty. \qquad (3.10)$$

Key: (1). with.

During the calculation of attenuation factor for high values y_i it is necessary to consider that when $y\to\infty$ function p(y) must satisfy the relationship/ratio

$$\lim_{y\to\infty}[p(y)-y]=0. \qquad (3.11)$$

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Therefore, representing function S(y') in the form

$$S(y')=\int_0^{y'}\sqrt{y-t}\,dy+\int_0^{y'}\left[\sqrt{p(y)-t}-\sqrt{y-t}\right]dy,$$

we we see that with $y'\to\infty$ first term unlimitedly grows/rises (infinite part it is equal to $\frac{2}{3}y'^{3/2}-t\sqrt{y'}$.) and the second approaches final limit, if difference p(y)-y vanishes sufficiently rapidly [for example in the manner that for function p(y), determined by formulas (3.18) and (3.19)].

Let us introduce value ξ_0 as the limit

$$\xi_0=\lim_{y'\to\infty}\left[S(y')-2S_0-\frac{2}{3}y'^{3/2}-t\sqrt{y'}\right]. \qquad (3.12)$$

Utilizing the approximate equality

$$S(y')-2S_0=\frac{2}{3}y'^{3/2}-t\sqrt{y'}-\xi_0,$$

valid at the high values of y', and replacing in the denominator of formula (3.02) for value $\sqrt{p(y')-t}$ on $\sqrt{y'}$, we we obtain attenuation factor in the form

$$V(x, y', y) = \sqrt[4]{\frac{x^2}{k}}\, e^{i\frac{2}{3}y'^{3/2}}\, V_1(\zeta, y), \tag{3.13}$$

where

$$V_1(\zeta, y) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{\pi}} \int\limits_C e^{i\zeta t}\, \Psi(t, y)\, dt \tag{3.14}$$

and

$$\Psi(t, y) = \frac{e^{i\xi_0} \sin S(y)}{\sqrt[4]{p(y) - t}\, [\chi(\nu)\, e^{-2iS_0} - i\, e^{-\pi\nu}]}. \tag{3.15}$$

Function $V_1(\zeta, y)$ is connected with the attenuation factor by the same V formula as in the theory of normal radiowave propagation. As in this theory, is natural to name/call V_1 the attenuation factor of plane wave. Since subsequently we will calculate only V_1, then we will frequently call V_1 simply attenuation factor.

The entering V_1 variable/alternating ζ is equal to

$$\zeta = x - \sqrt{y'}. \tag{3.16}$$

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The geometric sense of value ζ follows from Fig. 2, where through T is designated the point at which the incident plane wave (or spherical wave from the distant source) concerns the earth's surface. Value ζ is connected with the angle $\vartheta = \angle TCP$ (P - observation point, C - center of the Earth) or with the distance corresponding to it on

earth/ground $s=a\vartheta$ with the relationships/ratios

$$\xi = m\vartheta = m\frac{s}{a}, \quad m = \left(\frac{ka}{2}\right)^{\frac{1}{3}}. \tag{3.17}$$

Let us note that the point of contact of tangency T corresponds to course of ray in homogeneous atmosphere.

The infinite duct/contour C in the plane by complex variable t, on which are taken the integrals for V and V_1, to a considerable degree is arbitrary and must be selected so that the calculation of integral on it it would be possible to produce with the smallest labor/work, in particular, so that the main section of integration would be as far as possible small. In this case duct/contour C must cover entire of the pole of integrand in the positive direction so that they would be located above the duct/contour C. As the duct/contour of integration most comfortably it proved to be to select the duct/contour, depicted in Fig. 3; the salient point of this duct/contour was arranged/located either when $t = p(y_j)$, or several more left (cf. the end/lead of paragraph 6).

Into integrand $\Psi(t, y)$ they enter, as can be seen from formulas (3.05) and (3.06), the integrals of form $\int \sqrt{p(y) - t}\, dy$ with different t and different integration limits, including of composite ones.

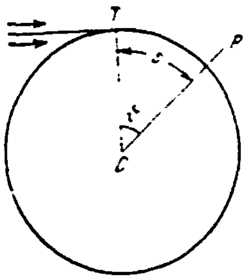

Fig. 2.

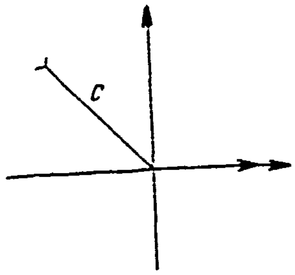

Fig. 3.

Fig. 2. Geometric sense of value $\zeta = m\theta$.

Fig. 3. Duct/contour C in plane of complex variable $\zeta - r(t)$.

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For facilitating the calculations of these integrals for the given refractive index M(h) was undertaken hyperbolic law (6.23) of chapter 15, whence function p(y) is obtained according to formula (3.03) in the form

$$p(y) = p(y_i) - \frac{(y - y_i)^2}{y + y_t}, \qquad (3.18)$$

moreover from relationship/ratio (3.11) it follows that

$$p(y_i) = y_t - 2y_i. \qquad (3.19)$$

By force of (3.19) in formula (3.18) are two parameters y_t and y_i.

moreover y_i is dimensionless spot height of inversion expedient to introduce also the special designation

$$Y = y_t - y_i. \qquad (3.20)$$

then

$$p''(y_i) = \frac{2}{Y}. \qquad (3.21)$$

Let us note that in the case of hyperbolic law equation (3.04) is quadratic equation with two roots of y_1 and y_2 flowing together when $t = p(y_i)$.

For the hyperbolic law integrals necessary to us are expressed as the elliptical integrals of the first and second kind. However, in the cases examined by us it proved to be more convenient to calculate these integrals with the aid of the expansions according to the degrees of parameter a^2, where

$$a^2 = \frac{t - p(y_i)}{4Y}. \qquad (3.22)$$

In these expansions, which contain also logarithmic components/terms/addends, sufficient to take several first terms, since the main section of integration on duct/contour C corresponds to the very low values of parameter a^2. Further terms of expansions are essential (at the high values of parameter Y undertaken by us, see the beginning of paragraph 5) only in such sections of the ducts/contours where entire/all integrand is already small.

In conclusion let us pause at the analytical continuation of

functions F(t, y', y) and $\Psi(t, y)$ to entire plane by complex variable t. The fact is that values S(y), S(y'), S_0 and $\chi(v)$, entering these functions, are initially determined only on the real axis when $t < p(y_i)$ $(v > 0)$, where for radicals $\sqrt[4]{p(y') - t}$ and $\sqrt[4]{p(y) - t}$ are taken arithmetical values. However, the knowledge of integrand when $t < p(y_i)$ proves to be sufficient only during the calculation according to the reflecting formula (paragraph 4). For calculating the contour integrals it is necessary to know integrand with composite t, which is reached with the aid of the analytical continuation.

In this case it is necessary to have in mind that at point $t = p(y_i)$ the precise functions F(t, y', y) and $\Psi(t, y)$ peculiarities do not have. Asymptotic expression (3.15) for function $\Psi(t, y)$ has, however, the singular points (branch point) with t=p(y) (for expressions $\sqrt[4]{p(y) - t}$ and S(y)) and with t=p(0) (for expressions S(y) and ξ_0). These singular points are obtained as a result of the fact that we use asymptotic expressions. In reality of branch point they are absent, since a precise integrand must be meromorphic. Therefore we is bypassed the "apparent singular points" from below, counting, for example, with t>p(y), that arc [p(y)-t] $=\pi$ and $\sqrt{p(y) - t} = i\sqrt{t - p(y)}$, moreover $\sqrt{t - p(y)} > 0$. This circuit/bypass is actually conditional, since formula (3.02) with t>p(y) is not applied in view of Stokes's so-called phenomenon. To disregard this phenomenon is possible only in the case when section t>p(y) gives a

small contribution to the value of contour integral, which in the cases examined by us occurs. The carried out by us control calculations with the aid of the functions of the parabolic cylinder (see Chapter 15 paragraph 3), which give a more precise asymptotic representation of subintegral of function $\Psi(t, y)$, they confirmed not only the qualitative, but also quantitative correctness of the results, obtained with the aid of formula (3.15).

Function $\Psi(t, y)$ has also pole, corresponding to the roots of equation (6.01). In the case when pole closely they approach the duct/contour of integration, it is necessary to bypass them from below.

4. Reflecting formula.

In the illuminated region natural to calculate attenuation factor according to the method of steady state, since this method gives transition to the laws of geometric optic/optics, applied sufficiently far from the horizon/level. To integral (3.14) the method of steady state can be used as follows. Let us represent integrand Ψ on the real axis in the form

$$\Psi = \frac{i}{2} \frac{e^{i\Omega(t)} - e^{i\Phi(t)}}{\sqrt{p(y) - t}\, \chi(t)\, (1 - \Lambda)}. \qquad (4.01)$$

where

$$\begin{aligned}\Omega(t) &= \xi_0 - S(y) - 2S_0 - \operatorname{arc}\chi(\nu),\\ \Phi(t) &= \xi_0 - S(y) - 2S_0 - \operatorname{arc}\chi(\nu) = \Omega(t) - 2S(y).\end{aligned} \quad (4.02)$$

$$-\operatorname{arc}\chi(\nu) = \nu\lg\nu - \nu - \operatorname{arc}\Gamma\left(\frac{1}{2} - i\nu\right) \quad (4.03)$$

and

$$\Lambda = \frac{i}{\chi(\nu)} e^{-\pi\nu + 2iS_0}. \quad (4.04)$$

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For the entire integrand in integral (3.14) it is possible to write with real t the expression

$$e^{i\lambda t}\psi = \frac{i}{2}\frac{e^{i\omega(t)} - e^{i\varphi(t)}}{\sqrt[4]{p(y) - t}\ \chi(\nu)\ (1 - \Lambda)}, \quad (4.05)$$

where

$$\begin{aligned}\omega(t) &= \xi t - \Omega(t),\\ \varphi(t) &= \xi t - \Phi(t).\end{aligned} \quad (4.06)$$

Since in this case ν is also real, first

$$\chi(\nu) = \sqrt{1 - e^{-2\pi\nu}} \quad (4.07)$$

and if $\nu > 0$, that

$$\Lambda = \frac{1}{\sqrt{1 - e^{2\pi\nu}}}. \quad (4.08)$$

Latter/last formula shows that when $\nu > 0$ $(t < p(y_i))$ absolute value Λ less than one (in particular, $\Lambda = \frac{i}{\sqrt{2}}$ when $\nu = 0$) and rapidly vanishes with increase ν. Therefore, if we search for the

point of steady state when $t < p(y_i)$, we can disregard the phase of denominator $1 - \Lambda$. Then the points of steady state t_1 and t_2 first and second term in the right side of formula (4.05) are obtained from equations

$$\omega'(t_1) = 0, \ \varphi'(t_2) = 0 \qquad (4.09)$$

or

$$\zeta = -\Omega'(t_1), \ \zeta = -\Phi'(t_2), \qquad (4.10)$$

moreover with the given ones ζ and y of value t_1 and t_2 are different.

Calculations show that the functions $-\Omega'(t)$ and $-\Phi'(t)$ have a maximum. Therefore we find two values of t_1 and two values t_2 (at least, if ζ is not too great). Should be taken only values of t_1 and t_2, which satisfy inequality $t < p(y_i)$ ($a^2<0$), since only for these values it is possible during the determination of the points of steady state to disregard the phase of denominator $1 - \Lambda$.

After finding points t_1 and t_2 we we can calculate integral (3.14), applying the method of steady state to each component/term/addend of formula (4.05). In this way we come to reflecting formula for the attenuation factor V_1:

$$V_1(\zeta, y) = \frac{e^{i\omega(t_1)}}{\sqrt[4]{p(y) - t_1}} \frac{A(t_1)}{\sqrt{-2\omega''(t_1)}} - \frac{e^{i\varphi(t_2)}}{\sqrt[4]{p(y) - t_2}} \frac{A(t_2)}{\sqrt{-2\varphi''(t_2)}}, \qquad (4.11)$$

where

$$A(t) = \frac{1}{\chi(\nu)(1-\Lambda)}. \qquad (4.12)$$

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The first member of reflecting formula (4.11) is direct wave, the second member - wave, reflected from the earth/ground. This formula has the same structure, that also the usual reflecting formula of geometric optic/optics; however, in it are represented the corrections, which appear in a strict examination of the passage of the waves through the layers, which adjoin the inversion point.

Let us note that with the decrease ζ of value t_1 and t_2 they decrease, and values ν corresponding to them increases. With sufficiently large positive ones ν is possible to count

$$\Lambda = 0, \ \chi(\nu) = 1, \ A = 1 \qquad (4.13)$$

and to utilize therefore for the functions $\Omega(t)$ and $\Phi(t)$ of the simpler expression

$$\begin{aligned} \Omega(t) &= \Xi_0 - S(y), \\ \Phi(t) &= \Xi_0 - S(y), \end{aligned} \qquad (4.14)$$

where

$$\Xi_0 = \xi_0 - 2S_0 = \lim_{y' \to \infty} \left[S(y') - \frac{2}{3} y'^{3/2} - t \sqrt{y'} \right]. \qquad (4.15)$$

With such simplifications reflecting formula (4.11)

converts/transfers into the usual reflecting formula, which escape/ensues from the laws of geometric optic/optics in heterogeneous atmosphere. The latter, similarly is applicable to the rays/beams, is sufficient to distant ones from the maximum rays/beams QO and QO" O' in Fig. 1b, it is more precise, to those rays/beams which have $\nu(t_1)$ and $\nu(t_2)$ — sufficiently large positive numbers. For quite maximum rays/beams, as it is easy to consider, has $\nu = 0$, and geometric optic/optics to them is not applied.

Returning to general reflecting formula (4.11), let us introduce for the maximum values $-\Omega'(t)$ and $-\Phi'(t)$ the designations

$$\zeta_1 = [-\Omega'(t)]_{\text{макс}}, \quad \zeta_2 = [-\Phi'(t)]_{\text{макс}}. \tag{4.16}$$

In view of formulas (4.02) is always fulfilled the inequality

$$\zeta_1 < \zeta_2. \tag{4.17}$$

Hence we see that to find the points of steady state t_1 and t_2 for both components/terms/addends in formula (4.05) is possible only with $\zeta<\zeta_1$. With $\zeta>\zeta_1$ the equation $\omega'(t)=0$ does not have real solution and direct wave is not expressed by first term of formula (4.11). Therefore value $\zeta=\zeta_1$ determines the horizon/level of direct waves (cf. paragraph 2). Analogously value $\zeta=\zeta_2$ determines the horizon/level of the waves, reflected from the earth/ground.

Physical sense ζ_2 consists in the fact that into the region $\zeta>\zeta_2$

the electromagnetic waves are drawn through only via diffraction; therefore $\zeta=\zeta_2$ there is boundary of the region of shadow.

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Physical sense ζ_1 lies in the fact that with $\zeta<\zeta_1$ is applicable reflecting formula (4.11), in consequence of which $\zeta=\zeta_1$ there is a boundary of the illuminated region. Region $\zeta_1<\zeta<\zeta_2$ is the intermediate region, included between both horizons/levels.

Since the maximum values of functions $-\Omega'(t)$ and $-\Phi'(t)$ attain near point $t=p(y_i)$, that value

$$\tilde{\zeta}_1=[-\Omega'(t)]_{t=p(y_i)},\quad \tilde{\zeta}_2=[-\Phi'(t)]_{t=p(y_i)}, \qquad (4.18)$$

they will be very close to the values, determined by formulas (4.16); we will show this based on examples in paragraph 5. Therefore the position of the horizons/levels it is possible approximately to determine from formulas of type (4.18)which is much simpler than to construct plotted functions $-\Omega'(t)$ and $-\Phi'(t)$, necessary with the use of formulas (4.16) For hyperbolic law (3.18) of formula (4.18) they lead to the expressions

$$\tilde{\zeta}_1=G_0-G(y);\quad \tilde{\zeta}_2=G_2-G(y), \qquad (4.19)$$

where

$$G_0 = \sqrt{y_1} - \frac{\sqrt{Y}}{2}\lg\frac{\sqrt{Y}+\sqrt{y_1}}{\sqrt{Y}-\sqrt{y_1}} + \frac{\sqrt{Y}}{2}\left[C_1 - \frac{1}{2}\lg(Y^3)\right]; \quad (4.20)$$

$$G(y) = -\sqrt{y_1 - y} - \sqrt{y_1} -$$
$$- \frac{\sqrt{Y}}{2}\left[\lg\frac{\sqrt{Y}+\sqrt{y_1-y}}{\sqrt{Y}-\sqrt{y_1-y}} - \lg\frac{\sqrt{Y}+\sqrt{y_1}}{\sqrt{Y}-\sqrt{y_1}}\right] \quad (4.21)$$

and

$$C_1 = C - 7\lg 2 - 4 = 1,429 \quad (4.22)$$

(C - Euler's constant).

Second formula (4.19) upon transfer to the usual (dimensional) coordinates gives formula for the horizon distance of the waves [formula (6.32) reflected of Chapter 15]. However, first formula (4.19) defines, as we already spoke, the horizon distance of direct waves.

Let us note in conclusion that reflecting formula (4.11) is applicable for calculating the attenuation factor V_1 almost to the very horizon/level of direct waves G_1.

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5. Numerical results in the dimensionless coordinates.

During the calculation of attenuation factor V_1 for the hyperbolic law of inversion we took the following numerical values of the parameters, entering function p(y) [formula (3.18)-(3.20)]:

Table 1.

№	y_i	h	Y	$p(y_i)$	$p(0)-p(y_i)$	ε
1	10.40	197.61	208.01	218.41	0.542	2.08
2	5.00	95.00	100.00	105.00	0.260	1.00
3	2.40	45.67	48.07	50.48	0.125	0.48
4	1.16	21.95	23.11	24.27	0.060	0.23

Functions p(y) at the selected values of the parameters are depicted in Fig. 4. The selection made by us allows with the defined M-profile/airfoil (see paragraph 7) to calculate propagation of four wavelengths which relate as by 1:3:9:27. In this case the first row Table 1 corresponds to shortest, and fourth - to longest wave.

In all cases we took $y=\frac{y_i}{5}$. i.e. was assumed height/altitude of one of the corresponding points the equal to one fifth height/altitude of the layer of inversion. Another point we took at the high altitude above the layer of inversion - such large that it is possible to use the attenuation factor $V_1(\zeta, y)$, connected with V according to formula (3.13).

The four curves calculated by us for the attenuation factor V_1 depending on variable/alternating ζ are given in Fig. 5 [1].

FOOTNOTE [1]. On all drawings, carried out on the logarithmic scale, along the logarithmic axis are plotted decimal (but not natural) logarithms. ENDFOOTNOTE.

Indices 1, 2, 3 and 4 in the curves they show, to what row table to I and what curve in Fig. 4 corresponds this curve for the attenuation factor. For each curve the point G_1 notes the position of the horizon/level of the direct waves, and point G_2 - position of the horizon/level of the waves, reflected from the earth/grouna. Points G_0 near the origins of coordinates, equipped with the same indices 1, 2, 3 and 4, determine the horizon/level (boundary of straight/direct visibility) with homogeneous atmosphere the corresponding values ζ_0 they are obtained from simple formula $\zeta_0 = \sqrt{y}$.

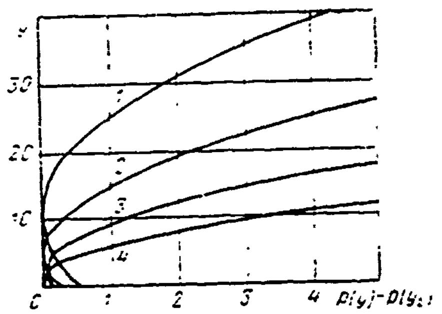

Fig. 4. Plotted functions $p(y_1)-p(y_2)$ for the values of the parameters Table 1.

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As is evident, into all four cases examined occurs the hyperdistant propagation, most strongly expressed, as this it was to be expected, in curve 1. Upon transfer to curves 2, 3 and 4 phenomenon of hyperdistant propagation monotonically weakens; however, on curved 4 with $\zeta \sim 5$ attenuation factor $|V|$ proves to be order 0.1, while with the same ζ and y, but in homogeneous atmosphere, $|V_1|$ takes values for four orders below ($|V_1| \sim 0.000013$).

Table 2 gives the values of function $|V_1|$ at the horizons/levels G_1 and G_2.

From it it is evident that the values of attenuation factor at

both horizons/levels G_1 and G_2 change within sufficiently wide limits - 3-3.5 times. For the comparison table 2 gives values $|V_1|$ at the horizon/level G_0 during the normal propagation and at the same values of y. The comparison of columns shows that during the normal propagation of the value of attenuation factor at the horizon/level they have because of the dependence on y much the same spread, as during anomalous propagation at the horizons/levels G_1 and G_2.

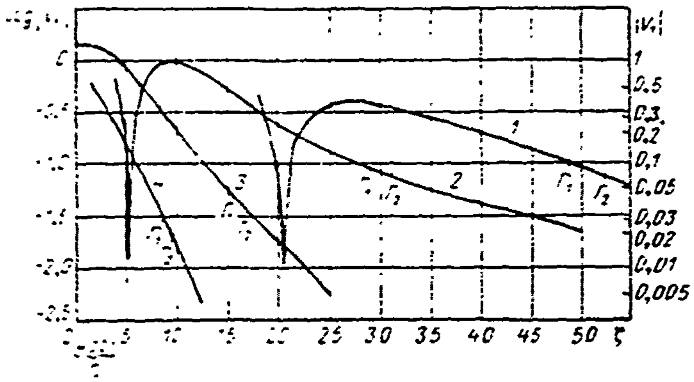

Fig. 5. Dependence of attenuation factor V_1 on ζ. The numbers of curves 1, 2, 3, 4 correspond to the numbers of rows Table 1 and to the numbers of the curves Fig. 4.

Table 2.

№	r_1	r_2	r_0	ν
1	0,06[illegible]	0,07[illegible]	0.24	2.08
2	0,095	0,080	0.19	1.0[illegible]
3	0,047	0,035	0.14	0.48
4	0,03[illegible]	0,023	0.083	0.23

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It is possible to note that at the horizons/levels G_1 and G_2 it does not occur a sudden change in the character of propagation. During the removal/distance from the source the attenuation factor

begins to monotonically decrease already in the illuminated region, it is more left both horizons/levels. This leads, in particular, to the fact that the factor of attenuation at the horizons/levels G_1 and G_2 proves to be on Table 2 2-4 times less than during the normal propagation at the horizon/level G_0. This behavior of attenuation factor is explained, apparently, by the fact that not only beyond the horizons/levels G_1 and G_2 but also are more left them they have a value the diffractive (are more precise, wave) phenomena, considered by reflecting formula (4.11) and which are not placed in the laws of geometric optic/optics.

For explaining the applicability of simple formulas (4.18)-(4.22) for calculating the horizon distance G_1 and G_2 let us compare in the cases examined by us the results given by them with the results according to formulas (4.16).

Table 3 shows that both formulas give very close numbers. Therefore for the practical calculations of horizon distance it is possible to use the simple formulas of Chapter 15.

Table 3.

№	[illegible]	[illegible]	[illegible]	[illegible]
1	49.11	49.11	52,26	52.26
2	26.56	26.56	39.74	39.74
3	16.08	15.99	17.52	17.50
4	8.67	8,45	9.52	9.50

6. Attenuation factor in a deep shadow. Series/row of deductions.

Attenuation factor in a deep shadow is conveniently investigated with the aid of the series/row of deductions which is obtained from integral (3.14) in the usual way (cf. chapter 14, paragraph 6). In order to obtain the series/row of deductions, it is first of all necessary to determine a precise position of poles of function $\Psi(t, y)$, i.e. the roots of the equation

$$1 - \Lambda = 0. \qquad (6.01)$$

These roots are located near duct/contour C (Fig. 3) or within it.

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If we designate

$$\Delta t = t - p(y_l). \qquad (6.02)$$

the value Δt for the roots found by us they are given in Table 4 whose first column shows the number of row in Table 1, and the second

- number of root for the present instance.

The position of the real parts three first roots relatively curved p(y) for the first case is represented in Fig. 6. We see that only the first root corresponds to the "seized" wave in the usual understanding, other two roots give the waves which from the point of view of geometric optic/optics easily would exceed the limits of the layer of inversion. However, these "being drawn through" waves have small fading and actively participate in the process of hyperdistant propagation. Let us recall that during the normal propagation t_1=1.17+i2.02, so that in this case the third wave attenuates 10 times more slowly than least damped wave under normal conditions for propagation. For the remaining cases all roots correspond to the "being drawn through" waves.

We convert equation (6.01) to the simple approximate form, which allows/assumes comparison with other theories of hyperdistant propagation. Let us begin from the "seized" waves which have almost real t, which lie between $p(y_i)$ and p(0) (as the first root in Table 4), and therefore negative values v. For $v<0$ we will assume

$$v=(-v)e^{i\pi}, \quad \lg v=\lg(-v)-i\pi. \qquad (6.03)$$

Table 4.

№	m	Δt_m	№	m	Δt_m
1	1	$0.10653 + i\ 0.00019$		2	$-0.1733 + i\ 0.3293$
	2	$-0.06364 + i\ 0.05523$	3	1	$-0.1036 + i\ 0.2236$
	3	$-0.1633 - i\ 0.2107$		2	$-0.1883 + i\ 0.6934$
	4	$-0.2495 - i\ 0.3913$	4	1	$-0.0852 + i\ 0.4661$
2	1	$-0.06335 - i\ 0.06518$		2	$-0.127[illegible] + i\ 1.1318$

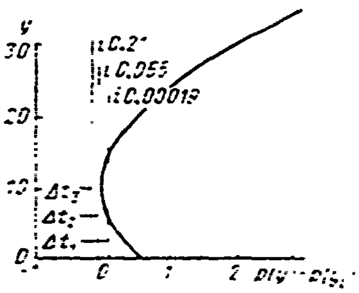

Fig. 6. Roots t_m corresponding to seized and unseized waves for data $p(y)-p(y_1)$

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Then together with (3.10) we will have

$$\chi(\nu)\to 1 \quad \text{with} \quad \nu\to-\infty \qquad (6.04)$$

and from formula (3.05) we obtain

$$S_0 = S_1 - \frac{i\pi}{2}\nu. \qquad (6.05)$$

where

$$S_1 = \int_0^{y_1} \sqrt{p(y) - t}\, dy, \qquad (6.06)$$

but y_1 indicates the smallest positive root of equation (3.04). Taking into account these formulas equation (6.01) takes the form

$$ie^{2iS_1} = \chi(\nu). \qquad (6.07)$$

If ν is great and is negative (strongly seized waves), then in view of relationship/ratio (6.04) we obtain the simpler equation

$$S_1 = \left(m - \frac{1}{4}\right)\pi; \quad m = 1, 2, \ldots \qquad (6.08)$$

corresponding to the known characteristic equation of the seized waves.

Let us visualize now that ν positively or is composite with the positive real part, i.e., $\mathrm{Re}\, t < p(y_0)$ or Re $\Delta t<0$. In this case value S_1 to determine with the aid of formula (6.06) is impossible at least because it is unknown, which of the composite roots of y_1 and y_2 one should take. However, turning formulas (6.05), we can always determine S with the aid of the relationship/ratio

$$S_1 = S_0 - \frac{i\pi}{2}\nu. \qquad (6.09)$$

and then from equation (6.01) we again obtain equation (6.07). With that being proper [i.e. corresponding to formula (6.03)] selection $\nu \to \infty$ we always have $\chi(\nu) \to 1$ (with exception of case $\arg \nu = -\frac{\pi}{2}$).

Since, furthermore, $\chi(0)=\sqrt{2}$, that for the "being drawn through" waves it is possible to count in the first, roughest, approximation/approach $\chi(v)=1$, and we obtain equation (6.08).

Let us note that simplified equation (6.08) usefully also for the normal propagation when in relationship (6.06) we obtain $p(y)=y$ and $y_1=t$. Thus, from (6.08) we obtain

$$t_m=\left[\frac{3}{2}\left(m-\frac{1}{4}\right)\pi\right]^{\frac{2}{3}}e^{i\frac{\pi}{3}}, \tag{6.10}$$

which approximately corresponds to the roots of characteristic equation for homogeneous atmosphere.

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For checking equation (6.08) we calculated (table 5) for the roots found by us value S_1 according to formula (6.09), and as a result were obtained these numbers:

Table 5.

№	m	$\left(m-\frac{1}{4}\right)\pi$	S_1	v
1	1	2.356	$2.326 - i\,0.001$	$-0.768 - i\,0.0014$
	2	5.498	$5.537 - i\,0.047$	$0.459 - i\,0.398$
	3	8.639	$8.646 - i\,0.009$	$1.178 - i\,1.519$
	4	11.781	$11.784 + i\,0.005$	$1.800 - i\,2.821$
2	1	2.356	$2.444 - i\,0.062$	$0.317 - i\,0.376$
	2	5.498	$5.501 - i\,0.011$	$0.867 - i\,1.646$
3	1	2.356	$2.315 + i\,0.011$	$0.360 - i\,0.776$
	2	5.498	$5.516 - i\,0.014$	$0.656 - i\,2.402$
4	1	2.356	$2.436 - i\,0.079$	$0.207 - i\,1.120$
	2	5.498	$5.499 - i\,0.007$	$0.319 - i\,2.718$

Thus, calculating S_1 for the obtained root, we with the aid of approximate relationship/ratio (6.08) can ascribe to it number m.

Fig. 7 depicts the attenuation factor in a deep shadow, calculated according to the series/row of deductions for the first case. Fig. 7 shows that the first term of the series/row of deductions, which corresponds to pole t_1, determines attenuation factor only with $\zeta>150$, i.e., for the wave $\lambda=1$ cm with s>1000 km. Since the first term has negligible fading, then at such large distances the absolute value of attenuation factor will be almost constant - asymptote in Fig. 7 is almost horizontal. Let us note that in a deep shadow in Fig. 7 attenuation factor approaches an asymptote, completing dying oscillations. These oscillations are caused by the interference first and of the second "simple wave".

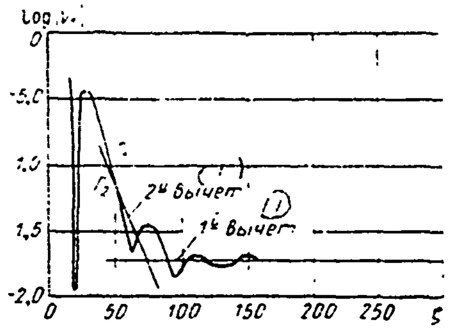

Fig. 7. Dependence on ζ that computed from the series/row of the deductions of attenuation factor V_1 in a deep shadow.

Key: (1). deduction.

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Thus, the first simple wave, which has the smallest fading, is very weakly excited by the wave, which falls on top to tropospheric waveguide, thanks to which this simple wave can be decisive only at the very large distances. Near the horizons/levels G_1 and G_2 primary meaning has the second and partly third term of the series/row of deductions. This phenomenon must have general character, since if simple wave is "seized" (see above) and barely it is drawn through from the layer of inversion (how and it is explained its negligible fading), then for reasons of reciprocity it is barely excited by the

emitters, arranged/located higher than the layer of inversion. The waves, which possess high fading, to the larger degree penetrate the space above the layer of inversion; therefore they are excited more strongly and play the dominant role near the horizons/levels.

Because of the noted fact the horizons/levels G_1 and G_2 actually/really define (although in the sufficiently approximate sense) the superstandard range of radio waves even during the strongly expressed superrefraction (as can be seen from Fig. 7).

Usually in the examination of hyperdistant propagation they are based on the series/row of deductions. In this case they assume that only seized waves $(\mathrm{Re}\,\Delta t_m > 0)$ can have a small fading. In fact and the waves "being drawn through" $(\mathrm{Re}\,\Delta t_m < 0)$ in a number of cases also attenuate weakly. Therefore waves several times longer than the "critical" wavelength λ_0, determined according to Bremmer [25], it is still capable of the hyperdistant propagation in the tropospheric waveguide.

Let us note in conclusion that, as showed calculations, several first terms of the series/row of deductions already make it possible to calculate attenuation factor by Poti up to the joining with the reflecting formula and they free thus from the calculations on the quadratures (cf. paragraph 3).

7. Numerical results for the specific case.

For facilitating the physical analysis of the numerical results, obtained by us in paragraph 5, we will examine the here corresponding to them specific case.

As an example we will take an M-profile/airfoil, depicted in Fig. 8, and will construct attenuation factor V_1 for the following waves: 1) 3.33 cm, 2) 10 cm, 3) 30 cm, 4) 90 cm. Graph/curve V_1 is given in Fig. 9. The numbers of curves Fig. 9 indicate the wavelengths enumerated here. Along the axis of abscissas we plot/deposit on the lower scale distance of s in the kilometers, and on the upper - angles ν in the degrees (cf. Figure 2). Along the axis of ordinates are deposited/postponed the common logarithms, and on the right scale are noted other values $|V_1|$.

Let us note that in our calculations is not taken into consideration the dispersion.

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We assume that an M-curve takes one and the same form for all four

waves for which Fig. 9 gives attenuation factor V_1.

M-curve in Fig. 8 is constructed according to the hyperbolic law

$$M(h)=M(h_i)+\frac{1}{a}\frac{(h-h_i)^2}{h-l}, \quad (7.01)$$

in which

$$M(h_i)=\frac{l-2h_i}{a}. \quad (7.02)$$

Hyperbolic law encompasses two parameters: h_i and l, having the dimensionality of height/altitude and connected with dimensionless constants y_i and y_l in formula (3.18) by the relationships/ratios

$$y_i=\frac{kh_i}{m}, \quad y_l=\frac{kl}{m},$$
$$m=\left(\frac{ka}{2}\right)^{\frac{1}{3}}, \quad (7.03)$$

moreover h_i is spot height of inversion or, which is the same, the height/altitude of radio duct. The height/altitude

$$H=h_i-l \quad \left(Y=\frac{kH}{m}\right), \quad (7.04)$$

as it is easy to show, is determined radius of curvature of M-curve at the inversion point.

In Fig. 9 distances are expressed in the kilometers. On the axis of abscissas is noted also the horizon/level G_0 during the propagation in homogeneous atmosphere.

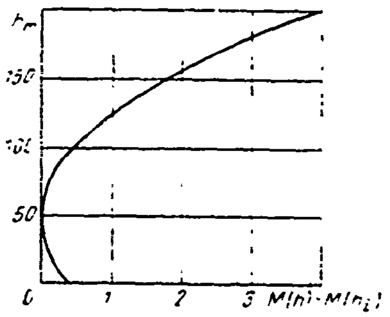

Fig. 8. Dependence on height/altitude h of the given refractive index (M-profile/airfoil) for the values of the parameters:

h_t = 46.5 m, l = [illegible] m, H = [illegible],
$M(h_t)$ = [illegible], $M(0) - M(h_t)$ = [illegible]

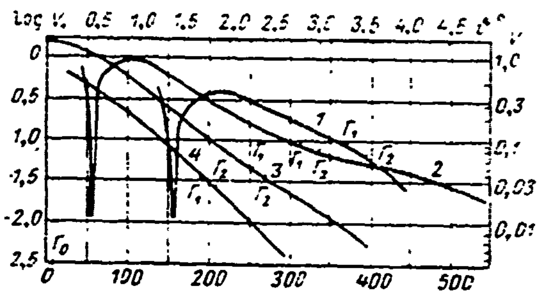

Fig. 9. Dependence of attenuation factor V_1 on distance of s (km) for wavelengths: curve 1 - 3.333 cm; curve 2 - 10 cm; curve 3 - 30 cm; curve 4 - 90 cm.

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This horizon/level is determined by spot height of observation h and does not depend on wavelength: let us recall that is everywhere

undertaken $h = \frac{h_i}{5}$. For each curved the point G_1 determines the position of the horizon/level of direct waves, and point G_2 - position of the horizon/level of the waves, reflected from the earth/ground (paragraphs 2 and 4). The horizons/levels G_1 and G_2 are changed with a change in the wavelength and therefore for each curve their.

In all cases it is possible to state/establish the phenomenon of hyperdistant radiowave propagation, which weakens with an increase in the wavelength. Taking into account a strong change in the wavelength upon transfer from one curve to another (wavelength they relate as by 1:3:9:27), one should recognize that near the horizons/levels the dependence of attenuation factor on the wavelength is comparatively weak.

Into formula for the horizon distance (cf. chapter 15, paragraph 6) the wavelength enters only under log sign. Therefore horizon distances form the arithmetical progression, if wavelengths as in Fig. 9, form geometric progression. In this case, however, the values of attenuation factor on both horizons/levels G_1 G_2 depends from the wavelength to the same degree, as during the normal radiowave propagation (cf. Table 2).

In view of these facts to identify the distance of propagation

of radio waves with the horizon distance of the straight/direct or reflected waves is necessary with certain precaution. It is possible to define superstandard range otherwise, for example as such distance at which the attenuation factor has absolute value of 0, 1, moreover on greater the distances of the value of attenuation factor it is still less. During the latter/last definition the "superstandard range" is included between the horizon distances G_1 and G_2 for curve 1 in Fig. 9, and for another curved this distance is less than the distance G_1; in the rough approximation and these four distances form the arithmetical progression, as can be seen from figure. Let us note that for evaluating the superstandard range according to value of 0.1 it usually proves to be sufficient to produce calculations with the aid of the reflecting formula of paragraph 4, only sometimes applying the extrapolation of curves obtained in this way.

The direct purpose of the reasonings of this chapter consisted, as we already spoke in paragraph 1, in checking of formulas for the superstandard range of radio waves, derived in the preceding/previous chapter. Above we showed that, introducing the horizons/levels of the straight/direct and reflected waves, it is possible to obtain the simple and demonstrative picture of hyperdistant radiowave propagation when the layer of inverse is present,. However, the distance of propagation only in the sufficiently rough sense can be identified with distance of one of the horizons/levels.

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The fact is that the decrease of attenuation factor (after the termination of oscillations in the illuminated region) begins earlier than we is reached first horizon. Therefore, as shown in paragraph 5, attenuation factor V_1 accepts at the horizon/level G_1 and G_2 of value, 2-4 times it is less than at the usual horizon/level G_0 during the propagation in homogeneous atmosphere. Furthermore, near the horizons/levels G_1 and G_2 attenuation factor decreases it goes without saying it is much slower than during the normal propagation.

All these reasons lead to the fact that the horizons/levels G_1 and G_2 during anomalous propagation characterize the superstandard range of radio waves more roughly than the horizon/level G_0 during the normal propagation. However, the possibility of applying the horizons/levels G_1 and G_2 for the approximate estimates of superstandard range does not produce the doubts, as is evident at least from the comparison of attenuation factor near the horizons/levels and of a deep shadow in Fig. 7.

One should emphasize that M-~~profic~~ profile selected by us has sufficiently weak inversion: difference $M(0) - M(h_i)$ does not exceed

several tenths. This inversion in certain cases can in practice remain unestablished. However, our calculations show that even this M-profile/airfoil strongly changes the character of radiowave propagation, leading to the hyperdistant propagation.

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Chapter 17.

Radiowave propagation on the surface tropospheric waveguide [1].

FOOTNOTE [1]. Foch, Weinstein and Belkina, 1958. ENDFOOTNOTE.

Chapter is dedicated to the theory of radiowave propagation between the corresponding points, which are located in the lowest layer of the atmosphere of inversion (in the tropospheric waveguide). To this case are applied the general formulas, derived in Chapter 15, moreover they are utilized both the functions of parabolic cylinder and Airy's function. With the aid of this procedure is conducted for a series/row of specific cases the calculation of the radiowave propagation of different length. From the obtained results it follows that hyperdistant (waveguide) propagation weakens with an increase in the wavelength only very slowly. Thus, for instance, wavelength can exceed the so-called "critical" wavelength by an order, and hyperdistant propagation will occur. At the end of the chapter is more precisely formulated the criterion of hyperdistant propagation.

Introduction.

Present Chapter is dedicated to the theory of radiowave propagation in the surface tropospheric waveguide (layer of inversion) on the assumption that both corresponding points are located within the waveguide. This propagation of radiowaves can be named/called intra-layer or intra-waveguide in contrast to the case (examined in Chapter 16), when one of the corresponding points is located highly above the layer of inversion.

To the investigation of radiowave propagation in the tropospheric waveguide is dedicated the series/row of the theoretical works (for example, see [23-25]); however, on this question there are still many vaguenesses. In this chapter we investigate intra-layer propagation, developing the methods, presented in Chapters 15 and 16.

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1. Basic formulas.

In this paragraph we will compare the basic formulas, obtained in Chapters 14, 15 and 16.

The attenuation factor of the V with the arbitrary dependence

refractive index of the atmosphere on the height/altitude is determined by means of the contour integral

$$V(x, y, y', q) = \sqrt{\frac{x}{\pi}} e^{-i\frac{\pi}{4}} \int_C e^{ixt} F(t, y, y', q)\, dt. \quad (1.01)$$

where duct/contour C covers in the positive direction entire of the pole of integrand (Chapter 14).

We are limited to value of q=∞, which corresponds to arbitrary polarization on the decimeter and shorter waves and to horizontal polarization on the waves of of longer. In this case function F(t, y, y', ∞)=F(t, y, y') takes the form (for y'>y):

$$F(t, y, y') = -\frac{1}{2i} f_1(y', t) \left[f_2(y, t) - \frac{f_2(0, t)}{f_1(0, t)} f_1(y, t) \right]. \quad (1.02)$$

Here y and y' essence the dimensionless heights/altitudes of the corresponding points:

$$y = \frac{kh}{m}, \quad y' = \frac{kh'}{m}; \quad (1.03)$$

value x in formula (1.01) is the nondimensional distance between these points, counted along the earth's surface:

$$x = \frac{ks}{2m^2}. \quad (1.04)$$

but parameter m in the latter/last formula has a value

$$m = \left(\frac{ka}{2}\right)^{1/3}, \quad (1.05)$$

where a is a radius of the Earth.

Functions f_1 and f_2 - the independent solutions of the differential equation

$$\frac{d^2f}{dy^2} + [p(y) - t]\, f = 0, \qquad (1.06)$$

moreover their Wronskian is assumed to be equal to -2i. Function p(y) is connected with the given refractive index M(h) with the relationship/ratio

$$p(y) = \frac{2m^2}{10^6} M(h) = 2m^2 \left(n - 1 + \frac{h}{a}\right) \qquad (1.07)$$

(n - the refractive index of air) and is its dimensionless analog.

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In Chapter 15 equation 81.06) integrated asymptotically on the assumption that p(y) has one minimum, i.e., that the given refractive index of the atmosphere has one inversion point. This character carries, in particular, the refractive index, depending on height/altitude according to the hyperbolic law

$$M(h) = M(h_i) + \frac{1}{a} \frac{(h - h_i)^2}{h - l} 10^6 \qquad (1.08)$$

(h_i — the height/altitude of inversion; l - parameter), to which corresponds the function

$$p(y) = p(y_i) - \frac{(y - y_i)^2}{y - y_l}. \qquad (1.09)$$

This function is accepted in all calculations both in the present Chapter and in Chapter 16.

Attenuation factor V, thus, in addition to dimensionless coordinates x, y, y', depend on parameters y_1 and y_2 (or Y= $y_2 - y_1$). characterizing function p(y).

As shown in Chapter 15, the high-altitude factors f_1 and f_2 are expressed as the functions of parabolic cylinder D_n as follows:

$$\left.\begin{aligned} f_1(y,t) &= c_1(\nu)\sqrt{2\frac{dy}{d\zeta}}\,g_1(\zeta),\\ f_2(y,t) &= c_2(\nu)\sqrt{2\frac{dy}{d\zeta}}\,g_2(\zeta),\end{aligned}\right\} \qquad (1.10)$$

where

$$\left.\begin{aligned} g_1(\zeta) &= D_{i\nu-\frac{1}{2}}\left(e^{-i\frac{\pi}{4}}\zeta\right),\\ g_2(\zeta) &= D_{-i\nu-\frac{1}{2}}\left(e^{i\frac{\pi}{4}}\zeta\right).\end{aligned}\right\} \qquad (1.11)$$

Expression for $D_n(z)$ in the form of series/row is given below [formula (2.05)].

Value ν is determined by the relationship/ratio

$$\nu = -\frac{1}{\pi}\int_{y_1}^{y_2}\sqrt{t-p(y)}\,dy. \qquad (1.12)$$

(y_1 and y_2 are the roots of equation p(y)-t=0). As a result of

$p''(y_i) > 0$ it is possible to accept, that $v > 0$ when $t < p(y_i)$; with t> $p(y_i)$ is obvious $v < 0$.

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Variables ζ and y are connected with the substitution

$$S(y) - S_0 = \int_0^{\zeta} \sqrt{\frac{1}{4}\zeta^2 + \nu}\, d\zeta, \tag{1.13}$$

where

$$S(y) = \int_0^{y} \sqrt{p(y) - t}\, dy; \tag{1.14}$$

$$S_0 = \frac{1}{2}\int_0^{y_1} \sqrt{p(y) - t}\, dy - \frac{1}{2}\int_0^{y_2} \sqrt{p(y) - t}\, dy, \tag{1.15}$$

and finally

$$\left.\begin{aligned} c_1(\nu) &= e^{-\frac{\pi\nu}{4} - i\frac{\pi}{8}} e^{i\left(\frac{1}{2}\nu - \frac{1}{2}\nu \lg \nu - S_0\right)}, \\ c_2(\nu) &= e^{-\frac{\pi\nu}{4} - i\frac{\pi}{8}} e^{-i\left(\frac{1}{2}\nu - \frac{1}{2}\nu \lg \nu - S_0\right)}. \end{aligned}\right\} \tag{1.16}$$

For the large positive ones ζ, which corresponds to heights/altitudes y, arranged/located higher than the inversion point y_t and sufficiently far from it, high-altitude factors allow/assume the asymptotic representation

$$f_1(y, t) = \frac{e^{i\frac{\pi}{4}}}{\sqrt[4]{p(y)-t}} e^{i(S-2S_0)},$$
$$f_2(y, t) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt[4]{p(y)-t}} e^{-i(S-2S_0)}. \quad (1.17)$$

Lower than inversion point and far from it (ζ is great and is negative) the asymptotic representation $f_1(y, t)$ and $f_2(y, t)$ takes the form

$$f_1(y, t) = \chi_1(\nu)\frac{e^{i\frac{\pi}{4}}}{\sqrt[4]{p(y)-t}} e^{i(S-2S_0)} - e^{-\pi\nu}\frac{e^{-i\frac{\pi}{4}}}{\sqrt[4]{p(y)-t}} e^{-iS},$$
$$f_2(y, t) = \chi_2(\nu)\frac{e^{-i\frac{\pi}{4}}}{\sqrt[4]{p(y)-t}} e^{-i(S-2S_0)} + e^{-\pi\nu}\frac{e^{i\frac{\pi}{4}}}{\sqrt[4]{p(y)-t}} e^{iS}, \quad (1.18)$$

where it is placed

$$\chi_1(\nu) = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{1}{2}-i\nu\right)} e^{-\frac{\pi\nu}{2}+i(\nu-\nu\lg\nu)},$$
$$\chi_2(\nu) = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{1}{2}+i\nu\right)} e^{-\frac{\pi\nu}{2}-i(\nu-\nu\lg\nu)}. \quad (1.19)$$

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Substituting expressions (1.18) into formula (1.02), we obtain

function F(t, y, y') in the case of intra-layer propagation (interesting to us) $(y \leqslant y' < y_1)$:

$$F(t, y, y') = \frac{i}{2} \frac{1}{\sqrt[4]{p(y) - t}\sqrt[4]{p(y') - t}} \times$$
$$\times \frac{[e^{iS(y')} - Ae^{-iS(y')}][e^{-iS(y)} - e^{iS(y)}]}{1 - A}, \qquad (1.20)$$

where

$$A = \frac{i}{\chi_1(\nu)} e^{-\pi\nu - 2iS_0}. \qquad (1.21)$$

Attenuation factor V was calculated from the series/row of the deductions

$$V(x, y, y') = 2\sqrt{\pi x} e^{i\frac{\pi}{4}} \sum_{m=1}^{\infty} R_m e^{ixt_m}. \qquad (1.22)$$

Through R_m is designated the deduction of function F(t, y, y') in m pole t_m. If we use approximation formula (1.20), then t_m is the m root of the equation

$$1 - A = 0, \qquad (1.23)$$

and deduction R_m takes the form

$$R_m = 2i \frac{1}{\sqrt[4]{p(y) - t_m}\sqrt[4]{p(y') - t_m}} \frac{\sin S(y)\sin S(y')}{\left[\frac{d}{dt}(1 - A)\right]_{t=t_m}}. \qquad (1.24)$$

The majority of numerical results in this work is obtained with the aid of formulas (1.22)-(1.24), moreover we were limited by three or four members of the series/row of deductions.

2. Calculation of attenuation factor with the aid of the functions of parabolic cylinder.

Asymptotic representations (1.18) of the high-altitude factors f_1 and f_2 on basis of which are obtained formulas (1.22)-(1.24), are valid, if ζ is great and is negative or it is located in certain sector around the negative real semi-axis.

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In the latter case absolute value ζ must be large. However, as can be seen from Table 1, in a number of cases interesting us appears the serious doubt of the applicability of asymptotic formulas, since value ζ proves to be on the modulus/module of the order of one or even less.

In order to explain this question, we calculated function V also with the aid of the high-altitude factors, expressed directly through the functions of parabolic cylinder [formula (1.10), (1.11)], but not through their asymptotic representations. In this case of pole t_m of integrand F(t, y, y') they are the roots of equation [compare (1.02) and (1.10)]

$$g_1(\zeta_0) = 0, \qquad (2.01)$$

where for brevity through ζ_0 is marked the value ζ, which corresponds y=0.

The remaining factors, entering $f_1(0, t)$, zero do not become. Deduction R_m is calculated from the formula

$$R_m = \frac{1}{2i} \frac{f_2(y, t_m) f_1(y', t_m) f_2(0, t_m)}{\left[\frac{d}{dt} f_1(0, t)\right]_{t=t_m}} \qquad (2.02)$$

[see also (1.10)]. Value ζ is found from the transcendental equation

$$\operatorname{sh} 2u + 2u = \frac{2}{\nu} [S(y) - S_0], \qquad (2.03)$$

which is obtained, if we calculate integral (1.13) with the aid of the substitution

$$\zeta = 2\sqrt{\nu} \operatorname{sh} u. \qquad (2.04)$$

Table 1.

[illegible]	(1) Номер полюса	ν	[illegible]	ζ	(2) Δt_m по формуле (1.23)	(2) Δt_m по формуле (1.25)
[illegible]	1	0.3[illegible]−i0.7[illegible]	[illegible]	−1.282−i0.041	−0.104−i0.2[illegible]	−0.113−i0.227
[illegible]	[illegible]	[illegible]	[illegible]	−0.726−i0.065	−0.085−i0.466	−0.10[illegible]−i0.4[illegible]
[illegible]	[illegible]	0.31[illegible]	[illegible]	−0.725−i0.246	−0.126−i1.12[illegible]	−0.12[illegible]−i0.1[illegible]
[illegible]	[illegible]	0.373−i0.6[illegible]	[illegible]	−1.56−i0.057	−0.148−i0.262	−0.156−i0.26[illegible]

Key: (1). Number of pole. (2). according to formula.

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The necessary root of equation is selected from the condition that at the real value of $t < p(y_i)$ value ζ must be negative real number, and from the considerations of continuity. For calculating functions (1.11) we used the series/row

$$D_n(z) = -\frac{2^{-\frac{n}{2}-1}}{\Gamma(-n)} e^{-\frac{1}{4}z^2} \sum_{m=0}^{\infty} \frac{(-1)^{m-1} 2^{\frac{m}{2}} \Gamma\left(\frac{m-n}{2}\right)}{m!} z^m. \quad (2.05)$$

It is first of all interesting to compare the roots of equation (1.23) and more precise equation (2.01). We bring in Table 1 of the value

$$\Delta t_m = t_m - p(y_i) \quad (2.06)$$

[see (4.02)], obtained according to formulas (1.23) and (2.01). Let us note that precisely value Δt_m is utilized actually during the calculations on the series/row of deductions and is determined,

therefore, the accuracy of results. Therefore we bring in all tables instead of very poles t_m of value Δt_m. (about the selection of values Y see paragraph 4). The coincidence of values Δt_m. of those calculated according to formulas (1.23) and (2.01), especially their alleged parts, can be considered satisfactory.

Fig. 1a and 1b gives curves [1] for the attenuation factor, calculated according to asymptotic formulas (1.23) and (1.24) (thin lines), also, with the aid of the functions of parabolic cylinder of formulas (2.01) and (2.02) (thick lines).

FOOTNOTE [1]. On all drawings, carried out on the logarithmic scale, along the logarithmic axis are plotted decimal (but not natural) logarithms. ENDFOOTNOTE.

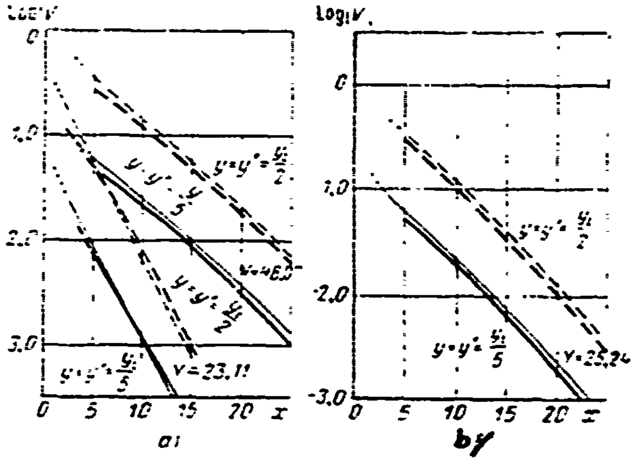

Fig. 1. Attenuation factor, calculated according to the formulas with the functions of parabolic cylinder (thick line and thick dotted line) and of the asymptotic formulas (thin line and thin dotted line): a) the I series; b) the II series, Table 3.

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From the comparison of these curves it is evident that both methods give even closer results, than it was possible to expect on the basis Table 1, especially for Y=23.11. Therefore the following presentation is based on calculations according to the asymptotic formulas as simpler, with exception of one case, examined in paragraph 3.

Analogous calculations we conducted also for the case, investigated in Chapter 16, moreover it turned out that the

asymptotic formulas were applicable there approximately/exemplarily with the same accuracy, as during the intra-layer propagation.

3. Calculation of high-altitude factors with the aid of the Airy's functions.

In paragraph 8 of Chapter 14 the derived approximation formulas for the high-altitude factors of the weakly damped "seized" waves, for which the pole corresponding to them is arranged/located between values $p(y_i)$ and p(0); high-altitude factors are expressed through the Airy's functions u(x) and v(x). In our case to the "seized" waves correspond the first of pole for the curves with Y=208.01 and Y=109.20 (see Table 4 and Fig. 5 and 6), and deductions in them can be calculated according to the formulas indicated.

The use of these formulas is caused by the fact that for some values of y, for example $y = y_i 2$, value $p(y)-t_1$ for the mentioned cases proves to be small (see Fig. 5 and 6)that is made itself impossible the use of asymptotic formulas of paragraph 1. However, the use/application of series/row (2.05) in this case is difficult, since in the cases of value ζ in question they are great and series/row converges very slowly.

The approximation formulas for the high-altitude factors of

weakly damped waves in case $y < y_i$ interesting us take the form

$$\left.\begin{aligned} f_1(y, t) &= 2c_1(\nu)\, e^{-\pi\nu}\psi_1(y, t), \\ f_2(y, t) &= 2c_2(\nu)\, e^{-\pi\nu}\psi_2(y, t), \end{aligned}\right\} \qquad (3.01)$$

where

$$\left.\begin{aligned} \psi_1(y, t) &= \sqrt[4]{\frac{\xi_1}{t - p(y)}}\left[v(\xi_1) + \frac{i}{4} e^{2\pi\nu} u(\xi_1)\right], \\ \psi_2(y, t) &= \sqrt[4]{\frac{\xi_1}{t - p(y)}}\left[v(\xi_1) - \frac{i}{4} e^{2\pi\nu} u(\xi_1)\right], \end{aligned}\right\} \qquad (3.02)$$

but ξ_1 is determined from the relationships/ratios

$$\left.\begin{aligned} \int_{y_1}^{y} \sqrt{t - p(y)}\, dy &= \frac{2}{3}\xi_1^{3/2} \qquad (y > y_1), \\ \int_{y}^{y_1} \sqrt{p(y) - t}\, dy &= \frac{2}{3}(-\xi_1)^{3/2} \qquad (y < y_1), \end{aligned}\right\} \qquad (3.03)$$

moreover through y_1 it is designated less of the roots of the equation

$$p(y) - t = 0. \qquad (3.04)$$

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The pole of integrand in this case are the roots of the equation

$$v(\xi_0) + \frac{i}{4} e^{2\pi\nu} u(\xi_0) = 0, \qquad (3.05)$$

where through ξ_0 is marked the value ξ, which corresponds y=0.

Finally, deduction R_1 is equal to

$$R_1 = -\frac{2ie^{-2\pi\nu}\psi_1(y,\ t_1)\psi_1(y',\ t_1)\psi_2(0,\ t_1)}{\left[\frac{d}{dt}\psi_1(0,\ t)\right]_{t=t_1}}. \qquad (3.06)$$

High-altitude factors can be represented with the aid of the Airy's functions when value $e^{2\pi\nu}$ is low. This condition is satisfied, if $\mathrm{Re}\,\nu$ is negative and is not small $(-\mathrm{Re}\,\nu \gtrsim 1)$. The latter occurs for poles t_1 with Y=208.01 and Y=109.20 (Table 2), therefore, this method for calculating the deductions in these poles is applicable.

It is possible to show that if ξ_1 is great and is negative, then formula (3.06) for remainder converts/transfers in (1.24). Thus, formula (1.24) is knowingly applicable, if ξ_1 is a large negative number (or it lies/rests at certain sector around the negative real semi-axis and great in the absolute value). But if ξ_1 is small or positive, then asymptotic formula (1.24) becomes unsuitable.

This is confirmed by Table 2. Both with Y=208.01 and with Y=109.20 to values $y = y_i/5$ correspond ν and ξ_1, having sufficiently large negative real parts, and the deductions, obtained according to formulas (1.24) and (3.06), prove to be close. To values $y = y_i/2$ with the same ν correspond small ξ_1 (but with Y=109.20 real part ξ_1 is already positive), and deductions, calculated according to formulas (1.24) and (3.06), sharply they differ. In this case the deductions must be calculated according to formula (3.06), and formula (1.24) is unsuitable.

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Thus, if value p(y)-t is low (ξ_1 is small), to calculate high-altitude factors according to formulas (1.18) is impossible, and it is necessary to use formulas (3.01)-(3.03). Let us note that the roots of equations (3.05) and (1.23) virtually coincide.

The use/application of Airy's functions makes it possible also to investigate the height/altitude of the layer, in which is propagated the seized wave. As can be seen from formulas (3.03), real part ξ_1 is negative only with $y<y_1$, where y_1 is a smaller root of equation (3.04). With $y>y_1$ value Re ξ_1 becomes positive and with increase in y increases (value Im ξ_1 always remains very small, since for the seized waves the alleged part t_1 is negligible). But with the large positive arguments function $v(\xi_1)$ exponentially decreases, and function $u(\xi_1)$ although increases, cannot become more than $e^{-\pi v}$ $(v<0)$ and therefore it is extinguished by factor $e^{2\pi v}$ (compare equations (3.03) and (1.12) and the asymptotic representations of Airy's functions in addition 2). Consequently, function $\psi_1(y, t)$, and means, and deduction R_1 after transition through $y=y_1$ rapidly decreases.

The dimensionless height/altitude y_1 it is possible, on the strength of what has been said above, to name/call the effective height/altitude of this seized wave, which corresponds to pole t_1, and in the first approximation, to consider that the field of this wave occupies the layer

$$0 < y < y_1. \tag{3.07}$$

but out of this layer it is negligibly small. In reality height/altitude $y=y_1$ is not the completely clear boundary of layer, but near this height/altitude occurs the smooth, but rapid weakening of the high-altitude factors (and thus, the fields of wave).

The noncaptured waves (y_1 is composite) do not possess the property indicated, and their field is distributed in entire layer of inversion and even it is higher.

4. Intra-layer propagation of waves (numerical results in the dimensionless coordinates).

Attenuation factor V depends on the dimensionless coordinates x, y, y' and, furthermore, from function p(y), which, in turn, is determined by parameters y_i and y_l. It is convenient to conduct the still dependent parameters, characterizing function p(y), namely:

$$Y = y_1 - y_0 \qquad (4.01)$$

$$p(y_2) = 2y_2 + y_0 \qquad (4.02)$$

$$p(0) - p(y_1) = \frac{y_2^2}{y_0}. \qquad (4.03)$$

Calculations were conducted for two series of the values of the parameters of function p(y), given in Table 3.

Table 2.

ν	ν	(1) Δf_1 по формуле (1.23)	(1) Δf_1 по формуле (3.05)
208,06	−0,768 − i 0,092	0,106 − i 0,092	0,101 − i 0,092
109,20	−1,851 − i 0,000	0,354 − i 0,094	0,317 − i 0,095

ν	λ_1	(1) R_1 по формуле (1.23)	(1) R_1 по формуле [illegible]
$\frac{1}{5}\nu_i$	−1,4[illegible] − i 0,0[illegible]2	0,10[illegible] − i 0,00[illegible]	0,10[illegible] − i 0,0[illegible]
$\frac{1}{2}\nu_i$	−0,241 − i 0,002	0,342 − i 0,001	0,142 − i 0,004
$\frac{1}{5}\nu_i$	−1,196 − i 0,092	0,303 − i 0,0[illegible]	0,283 − i 0,0[illegible]2
$\frac{1}{2}\nu_i$	0,382 − i 0,092	−[illegible],102 − i 0,2[illegible]	0,0762 − i [illegible]

Key: (1). according to the formula.

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Table 3.

(1) Номер кривой	[illegible]	[illegible]	[illegible]	$r(\mu_i)$	$p(0) - p(y_i)$
			I серия (2)		
1	208.01	10.40	197.61	218.41	0.542
2	100.00	5.00	95.00	105.00	0.260
3	48.07	2.40	45.67	50.48	0.125
4	23.11	1.16	21.95	24.27	0.060
			II серия (2)		
1	109.20	10.40	98.80	119.60	1.095
2	52.50	5.00	47.50	57.50	0.526
3	25.24	2.40	22.84	27.60	0.252

Key: (1). Number of curve. (2). series.

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Curves $p(y) - p(y_i)$ are plotted/applied in Fig. 2a (I series) and 2b (II series), moreover curves are marked by the numbers under which the parameters corresponding to them are given in the tables.

The parameters of function p(y) have selected we so that with this M-profile/airfoil it is possible to calculate attenuation factor for four wavelengths, which relate as 1:3:9:27 (I series) or three wavelengths, which relate as by 1:3:9 (II series).

Let us note that the I series of the parameters was accepted also in chapter 16 where was examined the case when one of the corresponding points is located highly above the layer of inversion, and other - within the waveguide. For the latter we took

$y = y_i/5$, where y_i is spot height of inversion.

In present chapter the attenuation factor V is calculated for the intra-waveguide propagation, moreover the heights/altitudes of transmitter and receiver are everywhere selected equal to (y=y'). Calculations are produced for two cases:

$$y = y' = \frac{y_i}{5}, \tag{4.04}$$

$$y = y' = \frac{y_i}{2}. \tag{4.05}$$

Attenuation factor $|V|$ is depicted as the function of nondimensional distance of x, moreover along the axis of ordinates are deposited/postponed the values of common logarithm $|V|$. Fig. 3 gives the curves of the I series, in Fig. 4 - the curves of the II series. Unbroken curves relate to case (4.04), broken lines - to case (4.05). The curves of the I series are calculated according to four, while the curves of the II series - on three members of the series/row of deductions.

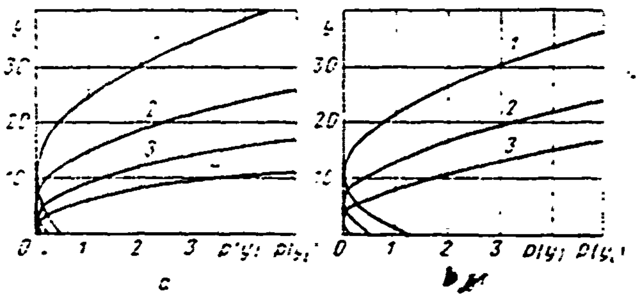

Fig. 2. Plotted functions $p(y)-p(y_i)$: a) the I series, b) the II series, table 3.

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Table 4 gives values Δt_m [formula (2.06)], entering all calculations and interesting, furthermore, also fact that they characterize the location of poles t_m relatively curved p(y). With $\mathrm{Re}\,\Delta t_m > 0$ we deal concerning "seized", very weakly damped wave.

This pole is in the first curved each series (Y=208.01 and Y=109.20, see Table 4, and also Fig. 5 and 6), that also explains the striking special feature/peculiarity of upper curves in Fig. 3 and 4. Instead of decreasing, as this seems natural for the attenuation factor, these curves increase. In fact, as can be seen from table 4, the alleged parts of the first poles in this case are very small, so that in the region of the applicability of one member of the

series/row of the deductions where there remains only "seized" wave, exponential factor virtually fading does not give. However, an increase in function V occurs because of factor $\sqrt{x}$ [see (1.22)]. The field of this wave carries cylindrical character, since is proportional not to 1/x, but $1/\sqrt{x}$. More demonstrative there will physically be here not function V, but function Ψ, by the specific relationship

$$V=2\sqrt{\pi x}e^{i\frac{\pi}{4}}\Psi, \qquad (4.06)$$

whence

$$\Psi=\sum_{m=1}^{\infty}R_m e^{ix t_m} \qquad (4.07)$$

[compare 1.22)].

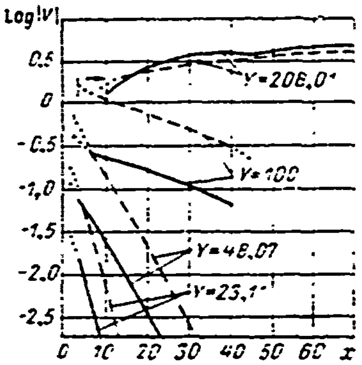

Fig. 3.

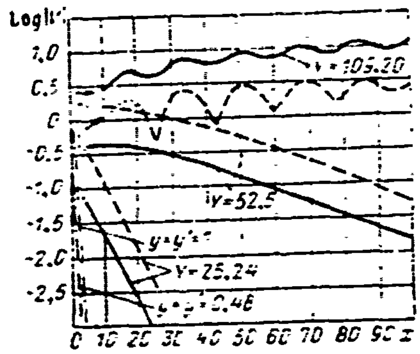

Fig. 4.

Fig. 3. Dependence of attenuation factor V on nondimensional distance of x (I series).

Fig. 4. Dependence of attenuation factor V on nondimensional distance of x (II series).

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It is the attenuation factor of cylindrical wave, which is spread in the surface tropospheric waveguide. Here seemingly there is reflecting layer (its height/altitude is approximately/exemplarily equal to y_1)beyond limits of which the seized wave barely is drawn through (compare the end/lead of paragraph 2). Therefore amplitude R_m of this wave with $y \gg y_1$ can with increase of y decrease, as it

takes place in Fig. 3 with Y=208.01 and 4 with Y=109.20; in all other cases the attenuation factor monotonically increases with increase in y.

As can be seen from Table 4 (see also Fig. 5 and 6), in cases of Y=208.01 and Y=109.20 the second of pole also have small alleged parts. Thus, the second simple wave for these cases attenuates although considerably more rapid than the first, it is nevertheless very slow. Therefore the oscillations of attenuation factor, which are obtained as a result of the interference of the first and second simple waves, for long do not disappear.

Table 4.

(1) Номер кривой	γ	(2) Номер полюса m	$\Delta\zeta_m$
		I серия (3)	
1	[illegible]	1	0.1065 − i 0.0002
		2	−0.0636 − i 0.0552
		3	−0.1633 − i 0.2107
		4	−0.2495 − i 0.3913
2	[illegible]	1	−0.0534 − i 0.0652
		2	−0.1732 − i 0.3293
		3	−0.2544 − i 0.6323
		4	−0.3164 − i 0.9459
3	[illegible]	1	−0.1038 − i 0.2235
		2	−0.1553 − i [illegible]
		3	−0.2254 − i 1.1712
		4	−0.2509 − i 1.6578
4	[illegible]	1	−1.0852 − i [illegible]
		2	−[illegible] − i 1.1316
		3	−[illegible] − i 1.84[illegible]
		4	−[illegible] − i 2.5[illegible]
		II [illegible] (3)	
1	[illegible]	1	0.3541 − i 0.000004
		2	−0.0962 − i 0.0150
		3	−0.1618 − i 0.1669
2	32.5[illegible]	1	−0.0497 − i 0.0492
		2	−0.2272 − i 0.3689
		3	−0.3529 − i 0.7653
3	25.24	1	−0.1484 − i 0.2621
		2	−0.2635 − i 0.8820
		3	−0.3146 − i 1.5143

Key: (1). Number of curve. (2). Number of pole m. (3). series.

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Fig. 7 and 8 well show damping oscillations of a function ψ around the almost horizontal asymptote, which is the first term of the series/row of deductions (4.07).

Let us note that in the case when one of the corresponding points is located highly above the layer of inversion (chapter 16), the first simple wave which slowly attenuates and provides therefore hyperdistant propagation, it is excited very weakly. Because of this it determines field only far in the shadow when several following simple waves, which are decisive for the field near the horizon/level, manage to damp. However, in the case of intra-layer propagation the first simple wave (Y=208.01 and Y=109.20) not only greatly weakly attenuates, but also is excited with the large amplitude.

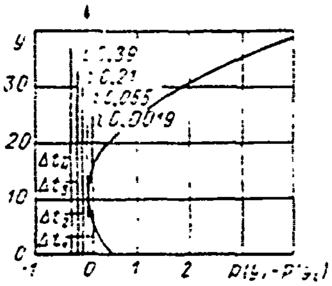

Fig. 5.

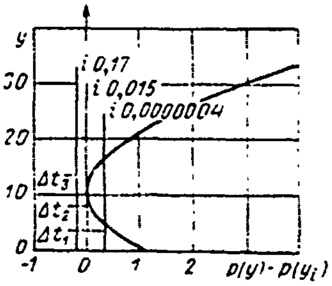

Fig. 6.

Fig. 5. Location of roots corresponding to seized and uncaptured waves, relative to graph/curve $p(y)-p(y_i)$ (I series).

Fig. 6. Location of roots corresponding to seized and uncaptured waves, relative to graph/curve $p(y)-p(y_i)$ (II series).

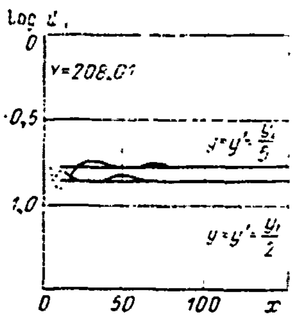

Fig. 7.

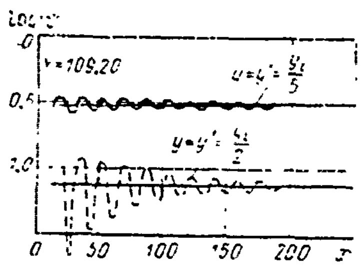

Fig. 8.

Fig. 7. Dependence of attenutation factor of cylindrial wave Ψ on nondimensional distance of x with Y=208.01.

Fig. 8. Dependence of attenutation factor of cylindrical wave Ψ on nondimensional distance of x with Y=109.20.

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At the remaining values of parameter Y there are no "seized" waves already. However, here in the composition of fields are sufficiently weakly damping simple waves, and field falls considerably slower than in the absence of inversion. For the comparison we give in Fig. 4 dot-dash line of the fields, obtained under the assumption of the absence of refractions for y=y'=1 and y=y'=0.48. They go Y=52.5 and Y=25.24, which give field at the same heights/altitudes, as dot-dash curves, but in the presence of inversion.

Simple dotted line in Fig. 3 and 4 noted those sections of curves in which used by us the members of the series/row of deductions give already sufficiently rough results, although even and accurate it is qualitative. These sections very closely approach the vertical lines, which note for each curve the geometric boundary of light/world and shadow without the refraction.

5. Numerical results for the specific case.

On the basis of the dimensionless curves, examined in paragraph 4, it is possible to obtain numerical results for the specific cases of the radiowave propagation of different length. For an example we

selected M-profile/airfoil, obtained from the first dimensionless functions p(y) I and of the II series (table 3, the I series, row 1 and table 3, the II series, row 1) of formula (1.07) (Fig. 9). Fig. 10 and 11 give the appropriate attenuation factors V for the wavelengths 3.33 cm (curves I), 10 cm (curves 2), 30 cm (curves 3) and 90 cm (curve of 4 Fig. 10). Height/altitude h, identical for the points of radiation/emission and reception/procedure, it is indicated in Fig. 10 and 11 in the meters. Distance s is given in the kilometers.

M-profile/airfoil I has the same height/altitude of inversion h_i, as M-profile/airfoil II, but doubly more strong inversion $M(0) - M(h_i)$. Consequently, we can compare attenuation factors for these profiles/airfoils, since Fig. 10 and 11 correspond to the identical heights/altitudes of the corresponding points/items and to identical wavelengths. Thereby we will obtain representation about the effect of inversion (at the same values h_i and λ) on the radiowave propagation.

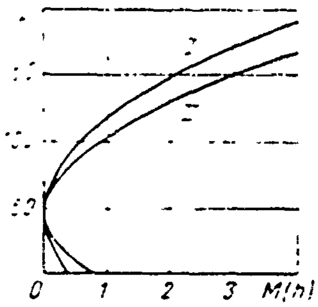

Fig. 9. Dependence of the given refractive index M on height/altitude h (M-profile/airfoil I and II).

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For curved 1 and 2 superstandard range proves to be greater in the layer with the more strong inversion, as this it was to be expected. However, for curved 3 occurs a somewhat unexpected effect: at short distances the curves barely differ, but from a certain point onwards field it proves to be more strongly in the case of weaker inversion. The physical sense of this result we will examine at the end of paragraph 6.

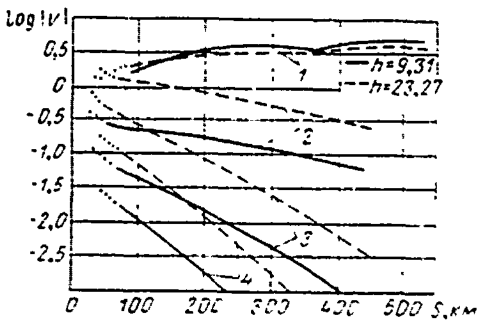

Fig. 10.

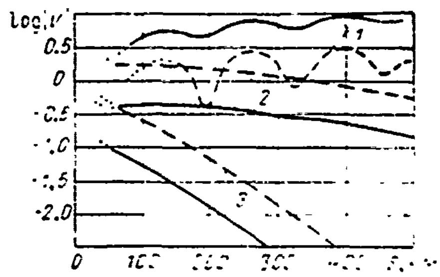

Fig. 11.

Fig. 10. Dependence of attenuation factor V on distance of s (in km) for wavelengths: 1 - 3.33 cm; 2 - 10.0 cm; 3 - 30.0 cm; 4 - 90 cm. Unbroken curve - for a height h=9.31 m. Dot-dash line - for a height h=23.27 m. (M-profile/airfoil I).

Fig. 11. Dependence of attenuation factor V on distance of s for lengths: 1 - 3.33 cm; 2 - 10.0 cm; 3 - 30 cm for heights/altitudes h=9.31 m and h=23.27 m (M-profile/airfoil II).

6. Attenuation of waves in the tropospheric waveguide.

As we saw above, attenuation factor V is represented in the form of the series/row of deductions (1.22). Complex numbers t_m determine the dependence of the separate terms of the series/row of deductions (or "simple" waves) on nondimensional distance of x. They are the roots of equation (1.23) or more precise equation (2.01). We investigate in more detail the dependence of numbers t_m on different factors.

As it was shown in the paragraph of 6 chapters 16, equation (1.23) approximately is reduced to the simpler equation

$$S_1 = \left(m - \frac{1}{4}\right)\pi. \qquad (6.01)$$

Here m is number of this root $t = t_m$, and S_1 - integral

$$S_1 = \int_0^{y_1} \sqrt{p(y) - t}\, dy, \qquad (6.02)$$

where through y_1 is designated the smaller root of the equation

$$p(y) - t = 0. \qquad (6.03)$$

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Formulas (6.01) and (6.02), as it is easy to show, remain applicable when the roots of equation (6.03) are composite, if we by y_1 understand root with the positive alleged part. With their aid it is

possible to approximately find roots τ_m, corresponding both weakly to the damped and strongly damped simple waves in series/row (1.22). We will use these formulas in order to explain, on what parameters depends the attenuation of simple waves. This will make it possible to obtain representation about what parameters of inversion M-profile/airfoil determine first of all the superstandard range of radio waves and under what conditions occurs hyperdistant propagation.

At first glance seems that the basic parameters are height/altitude inversions h_i and complete increase $M(0) - M(h_i)$ in the layer of inversion. Actually/really, they determine the so-called "critical wavelength" λ_m for the m simple wave in the surface tropospheric waveguide:

$$\lambda_m = \frac{10^{-3}}{m - \frac{1}{4}} h_i \sqrt{2[M(0) - M(h_i)]} \qquad (6.04)$$

(see Bremmer's book [25]).

In reality the assumption about the primary meaning of these parameters is incorrect. In order to study this question, let us introduce instead of y and p(y) the new given variable/alternating:

$$z = \frac{y}{y_i} = \frac{h}{h_i}; \quad q(z) = 4\frac{p(y) - p(y_i)}{p(0) - p(y_i)} = 4\frac{M(h) - M(h_i)}{M(0) - M(h_i)}, \qquad (6.05)$$

so that function q(z) always satisfies the relationships/ratios

$$q(0) = 4, \quad q(1) = 0. \qquad (6.06)$$

Instead of by the variable/alternating t let us introduce value

$$\tau = 4\frac{t - p(y)}{p(0) - p(y_i)}. \qquad (6.07)$$

Then equation (6.01) will take the form

$$\int_0^{z_1} \sqrt{q(z) - \tau}\, dz = \frac{\left(m - \frac{1}{4}\right)\pi}{\sqrt{G}}, \qquad (6.08)$$

where

$$G = \frac{y_i^2}{4}[p(0) - p(y_i)] = \frac{k^2 h_i^2}{2} 10^{-6}[M(0) - M(h_i)], \qquad (6.09)$$

and z_1 is a root of the equation

$$q(z) - \tau = 0, \qquad (6.10)$$

that corresponding y_1.

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For this M-profile/airfoil with the inversion critical wavelength λ_m is determined from the relationship/ratio

$$\int_0^1 \sqrt{q(z)}\, dz = \frac{\left(m - \frac{1}{4}\right)\pi}{\sqrt{G_m}}, \qquad (6.11)$$

where G_m is value of G when $\lambda = \lambda_m$. If we assume

$$\beta = \int_0^1 \sqrt{q(z)}\, dz, \qquad (6.12)$$

for the majority of M-curves value β will be close to unity as a result of formulas (6.06). Thus, for M-profiles/airfoils examined above we have β=0.991 (I series) and β=0.983 (II series). From formulas (6.11) we obtain

$$\sqrt{G_m} = \frac{\left(m - \frac{1}{4}\right)\pi}{\beta}, \qquad (6.13)$$

whence with the aid of relationship/ratio (6.09) we find

$$\lambda_m = \frac{\beta 10^{-3}}{m - \frac{1}{4}} h_i \sqrt{2[M(0) - M(h_i)]}. \qquad (6.14)$$

This expression differs from (6.04) only in terms of the presence of factor β.

Name "critical wavelength" is introduced in the literature for following reasons. When $G > G_m$, i.e. when $\lambda < \lambda_m$, equation (6.08) has real root $\tau = \tau_m$, lying within the limits $0<\tau<4$. To this root in series/row (1.22) corresponds the undamped simple wave. When $G < G_m$ i.e. when $\lambda > \lambda_m$, equation (6.08) has the composite root which determines the damped simple wave.

We will subsequently use term "critical wavelength", understanding it as the abbreviated/reduced designation of expression (6.14). However, at the critical wavelengths it does not occur any qualitative jump (see below), so that this designation is conditional.

Equation (6.08) can be written in the following form:

$$\int_0^{z_1} \sqrt{q(z) - \tau}\, dz = \beta \sqrt{\frac{G_m}{G}} = \beta \frac{\lambda}{\lambda_m}. \qquad (6.15)$$

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Hence it is apparent that for this M-profile/airfoil [and, therefore, for the fixed/recorded function q(z)] all roots τ must depend only on relation λ_m/λ. Instead of this relation to more conveniently use its common logarithm

$$l = \log\frac{\lambda_m}{\lambda} = \log\sqrt{\frac{G}{G_m}}. \quad (6.16)$$

Thus, if different M-curves have coinciding functions q(z), then the corresponding roots τ must be placed to one and the same curve

$$\tau = f(l). \qquad (6.17)$$

The drawn conclusions utilize equation (6.01) for roots t_m. Actually during the calculations we this equation did not use, since it is too rough; however, the values τ obtained of the more precise equations are approximately subordinated to these laws. Fig. 12 gives functions q(z) for two series of M-curves, examined by us above. These functions with 0<z<2 virtually coincide, although M-curves themselves noticeably differ (compare Fig. 9). Therefore values τ for all calculated earlier roots t_m (see Table 4) lie down near certain common curve (6.17). This is evident from Fig. 13, where the light small circles plotted/applied values Im τ, which correspond to roots t_m in Table 4 (series I), and by black small circles - values Im τ,

which correspond to roots t_m. to those refined according to the formulas of paragraph 2 (see Table 1); triangles noted points on Tables 4 (series II).

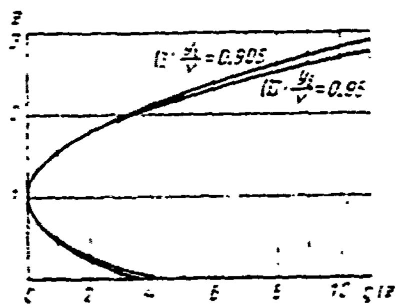

Fig. 12.

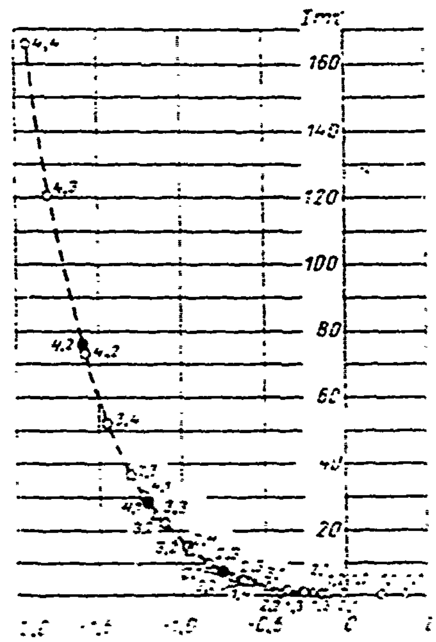

Fig. 13.

Fig. 12. M-profile/airfoil in new given variable/alternating [function q(z)] for I and II series.

Fig. 13. Alleged parts of roots τ of equation (6.15) depending on by variable/alternating 1 (6.16).

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Numerals about the small circles and the triangles indicate number by the curve p(y) (first numeral) and number of root m (second numeral).

Thus, for instance, 3.2 indicate the second root for third by the curve p(y). Let us recall that the alleged parts of numbers τ_m and l_m determine the attenuation of simple waves in series/row (1.22).

We see also that with l=0, i.e., when $\lambda = \lambda_m$, it does not occur any qualitative change in the attenuation of the m simple wave. This fading according to a strict theory is both when $\lambda < \lambda_m$. and when $\lambda > \lambda_m$ and sufficiently slowly it increases with an increase λ. Because of this hyperdistant propagation can to be observed (although into a somewhat weakened form) at the wavelength λ, by an order exceeding the greatest "critical" value λ_1. Actually/really, already curve 2 in Fig. 10 corresponds to wavelength λ, larger than λ_1, although hyperdistant propagation has the place even for the longer waves (curves 3 and even 4).

If function q(z) for the undertaken M-profile/airfoil strongly differs from the functions, depicted in Fig. 12, then shape of the curve (6.17) will qualitatively be the same, but numerical relationships/ratios will be completely different. Thus, assuming/setting

$$q(z) = 4\left(1 - \frac{z - \frac{z^n}{n}}{1 - \frac{1}{n}}\right). \qquad (6.18)$$

we with n=1/2 and n=1/5 we come to the cases, examined in works [23] and [24]. These functions q(z) are depicted in Fig. 14. They strongly

differ from the functions, plotted/applied in Fig. 12, and the values τ for them with the same 1 are obtained approximately/exemplarily by an order less than in Fig. 13. Latter/last confirmation follows from the figures of Hartree's article with co-authors [24], if we consider the relationships/ratios between the designations of Hartree and our designations.

Thus, Fig. 13 is not universal, and to calculate with its aid the attenuation of simple waves with any form of M-profile/airfoil is impossible; hypothesis about the fact that the hyperdistant propagation is determined only by the height/altitude of inversion h_i and by increase $M(0)-M(h_i)$, proves to be invalid.

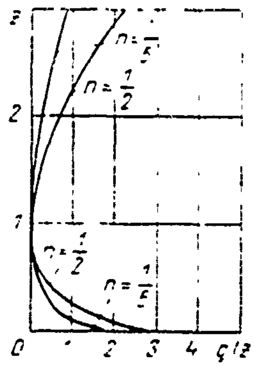

Fig. 14. Functions q(z) from (6.18) with n=1/2 and n=1/5. (M-profiles/airfoils, accepted in [23] and [24]).

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Hence it follows that there is at least one additional parameter, which characterizes M-curve and which has fundamental importance for the hyperdistant propagation. As this parameter is natural to take curvature of M-curve at the inversion point, i.e., value $M''(h_i)$, since this parameter substantially affects the infiltration of electromagnetic field from the layer of inversion.

If we proceed from the assumption that value $M''(h_i)$ is also the basic parameter, then instead of values q(z) and τ should be introduced the new given values

$$Q(z) = \frac{q(z)}{q''(1)}, \quad T = \frac{\tau}{q''(1)}. \quad (6.19)$$

where for the hyperbolic law of the inversion

$$q''(1) = 8\frac{\mu_i}{\gamma}, \qquad (6.20)$$

but for power law (6.18)

$$q''(1) = 4n. \qquad (6.21)$$

In Fig. 15 the curve A corresponds to the center lines, near which are grouped the values τ, given in Fig. 13. The curve B is constructed according to the values, undertaken from Hartree's article [24] (for the function (6.18) with n=1/5).

Comparing between themselves the curves A and B in Fig. 15, we see that two completely different types of M-profiles/airfoils (compare Fig. 12 and 14) give, if we use given values (6.19), not too differing values of the attenuation factors of simple waves in the surface tropospheric waveguide. Therefore given values (6.19) are more convenient for the comparison of the properties of different layers of inversion, than value (6.05) and (6.07). This result confirms the hypothesis made above about that which $M'(h_i)$ is also the basic parameter, which are determining the infiltration of electromagnetic energy from the layer of inversion. Let us note that real parts T for two types of inversion examined are distinguished much more strongly.

Since according to series/row (1.22), the reduction of the amplitude of the m simple wave is determined by the factor

$$e^{-\mathrm{Im}\,\xi_m x} = e^{-\varkappa_m s}. \tag{6.22}$$

where s is a horizontal distance betw-e :.e corresponding points, the more demonstrative representation about the attenuation of wave gives the coefficient

$$\varkappa_m = \frac{2\pi}{\lambda_m} 10^{-?} h_i^2 M^* (h_i)\, \Theta. \tag{6.23}$$

where value Θ is equal to

$$\Theta = \frac{\lambda_m}{l} \mathrm{Im}\, T. \tag{6.24}$$

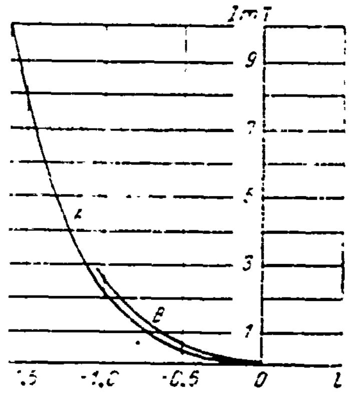

Fig. 15. Comparison of the alleged parts T (6.19) for M-profiles/airfoils of different types (A - type of the profile/airfoil Fig. 12; B - type of the profile/airfoil Fig. 14).

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This value is plotted/applied in Fig. 16 as function from l, moreover the curves A and B are constructed according to to the curves A and B Fig. 15. Formula (6.23) can be rewritten as follows:

$$\varkappa_m = 2\pi\left(m - \frac{1}{4}\right)10^{-3}\,\frac{[illegible]}{[illegible]}\,\Theta \qquad (6.25)$$

Since in formulas (6.23) and (6.25) on wavelength depends only Θ, then Fig. 16 directly indicates the character of the dependence of fading on the wavelength. First with an increase in the wavelength fading increases, but then it reaches maximum and begins to decrease. Tne latter is easy to understand physically, since on the propagation of sufficiently long waves the layer of inversion has a small effect,

and such radio waves are spread further beyond the horizon/level, than their length it is more, as it takes place, also, during the normal refraction.

The obtained results show that upon consideration of three parameters of M-curve $(h_i, M(0)-M(h_i)$ and $M''(h_i))$ the attenuation factors of simple waves for the strongly differing types of M-profiles/airfoils although are obtained one order, they differ quantitatively (in certain cases even 1.5-2 times, see Fig. 16). This means that to precise account for the effect of M-profile/airfoil on the propagation it is necessary to take into consideration even some parameters. This question needs supplementary investigation.

Propagation in the surface waveguide can be rated/estimated, comparing attenuation factor $\varkappa_1$ with the attenuation of the first simple wave in the theory of radiowave propagation during the normal refraction. This comparison makes it possible to explain, how this a propagation can be considered "hyperdistant". The criterion of hyperdistant propagation is somewhat indefinite; however, this uncertainty/indeterminancy is caused by the very nature of the phenomenon, which does not undergo abrupt changes with a continuous change in the wavelength and layer of inversion.

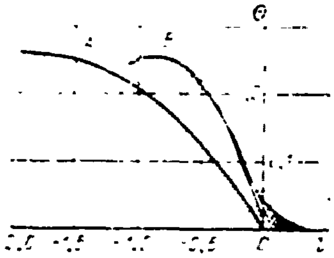

Fig. 16. Dependence on 1 attenuation factor Θ of simple wave (6.24) during the propagation in the tropospheric waveguide (M-profile/airfoil of types A and B).

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In calculations examined above in the second series of M-curves is undertaken doubly larger value $M(0)-M(h)$ at the same height/altitude of the layer of inversion, in consequence of which increased critical wavelengths (6.14). Generally speaking, short waves better are propagated in the waveguides with the larger critical wavelength (II series), than in the waveguides with the smaller critical wavelength (I series). However, in the second series increases curvature of M-curve at the inversion point, thanks to which is facilitated the infiltration of electromagnetic energy of longer waves from the layer of inversion. As a result for the shorter waves distant propagation proves to be best in the first series, whereas for the longer waves position will be reverse/inverse (curve

3 in Fig. 11 corresponds to a somewhat worse propagation up to the greater distances than curve 3 in Fig. 10).

During the calculation of field in the layer of inversion it is most convenient to use the series/row of deductions (1.22). Physically this means that more distant and hyperdistant radiowave propagation between two corresponding points, which are located within the layer of inversion, can be considered as the transmission of cylindrical waves along the peculiar transmitting "line" - to the layer of inversion. The attenuation of these waves is caused by radiation losses, their amplitudes R_n depend on field distribution according to the height/altitude.

The results obtained above illuminate the series/row of the laws, which relate to the transmission of radio waves on the surface tropospheric waveguide.